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INVARIANT DIFFERENTIAL OPERATORS ON THE MINKOWSKI-EUCLID SPACE

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Abstract. For two positive integers m and n, let $\mathcal{P}_n$ be the open convex cone in $\mathbb{R}^{n(n+1)/2}$ consisting of positive definite $n \times n$ real symmetric matrices and let $\mathbb{R}^{(m,n)}$ be the set of all $m \times n$ real matrices. In this paper, we investigate differential operators on the non-reductive homogeneous space $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ that are invariant under the natural action of the semidirect product group $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ on the Minkowski-Euclid space $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$. These invariant differential operators play an important role in the theory of automorphic forms on $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ generalizing that of automorphic forms on $GL(n,\mathbb{R})$.

1. Introduction

Let

$$\mathcal{P}_n = \left\{ Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY > 0 \right\}$$

be the open convex cone of positive definite symmetric real matrices of degree n in the Euclidean space $\mathbb{R}^{n(n+1)/2}$, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l and ${}^tM$ denotes the transpose matrix of a matrix M. Then the general linear group $GL(n,\mathbb{R})$ acts on $\mathcal{P}_n$ transtively by

$$g \cdot Y = gY\,{}^tg, \qquad g \in GL(n,\mathbb{R}),\ Y \in \mathcal{P}_n. \tag{1.1}$$

Therefore, $\mathcal{P}_n$ is a symmetric space which is diffeomorphic to the quotient space $GL(n,\mathbb{R})/O(n)$, where $O(n)$ denotes the orthogonal group of degree n. A. Selberg [10] investigated differential operators on $\mathcal{P}_n$ invariant under the action (1.1) of $GL(n,\mathbb{R})$ (cf. [7, 8]).

Let

$$GL_{n,m} = GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$$

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be the semidirect product of $GL(n,\mathbb{R})$ and the abelian additive group $\mathbb{R}^{(m,n)}$ equipped with the following multiplication law

$$(g,\lambda)\cdot(h,\mu)=\big(gh,\lambda\,{}^t h^{-1}+\mu\big),$$

where $g,h\in GL(n,\mathbb{R})$ and $\lambda,\mu\in\mathbb{R}^{(m,n)}$. Then we have the *natural action* of $GL_{n,m}$ on the non-reductive homogeneous space $\mathcal{P}_n\times\mathbb{R}^{(m,n)}$ given by

$$(g,\lambda)\cdot(Y,V)=\big(gY\,{}^t g,\,(V+\lambda)\,{}^t g\big), \tag{1.2}$$

where $g\in GL(n,\mathbb{R}),\ \lambda\in\mathbb{R}^{(m,n)},\ Y\in\mathcal{P}_n$ and $V\in\mathbb{R}^{(m,n)}$.

For brevity, we set $\mathcal{P}_{n,m}=\mathcal{P}_n\times\mathbb{R}^{(m,n)}$ and $K=O(n)$. Since the action (1.2) of $GL_{n,m}$ is transitive, $\mathcal{P}_{n,m}$ is diffeomorphic to $GL_{n,m}/K$. We observe that the action (1.2) of $GL_{n,m}$ generalizes the action (1.1) of $GL(n,\mathbb{R})$.

The significance in studying the non-reductive homogeneous space $\mathcal{P}_{n,m}$ may be explained as follows. Let

$$\Gamma_{n,m}=GL(n,\mathbb{Z})\ltimes\mathbb{Z}^{(m,n)}$$

be the arithmetic subgroup of $GL_{n,m}$, where $\mathbb{Z}$ is the ring of integers. The arithmetic quotient $\Gamma_{n,m}\backslash\mathcal{P}_{n,m}$ may be regarded as the universal family of principally polarized real tori of dimension mn (cf. [14]). We propose to name the space $\mathcal{P}_{n,m}$ the *Minkowski-Euclid space* since it was H. Minkowski [9] who found a fundamental domain for $\mathcal{P}_n$ with respect to the arithmetic subgroup $GL(n,\mathbb{Z})$ by means of the reduction theory. In this setting, using the invariant differential operators on $\mathcal{P}_{n,m}$, we can develop a theory of automorphic forms on $\mathcal{P}_{n,m}$ generalizing that on $\mathcal{P}_n$.

The aim of this paper is to study differential operators on $\mathcal{P}_{n,m}$ that are invariant under the action (1.2) of $GL_{n,m}$. This paper is organized as follows. In Section 2, we review differential operators on $\mathcal{P}_n$ invariant under the action (1.1) of $GL(n,\mathbb{R})$. In Section 3, we investigate differential operators on $\mathcal{P}_{n,m}$ invariant under the action (1.2) of $GL_{n,m}$. For two positive integers m and n, let

$$S_{n,m}=\Big\{(X,Z)\mid X={}^tX\in\mathbb{R}^{(n,n)},\ Z\in\mathbb{R}^{(m,n)}\Big\}$$

be the real vector space of dimension $\frac{n(n+1)}{2}+mn$. From the adjoint action of the group $GL_{n,m}$, we have the *natural action* of the orthogonal group $O(n)$ on $S_{n,m}$ given by

$$k\cdot(X,Z)=(k\,X\,{}^tk,Z\,{}^tk),\quad k\in O(n),\ (X,Z)\in S_{n,m}. \tag{1.3}$$

The action (1.3) of $K=O(n)$ induces canonically the representation σ of $O(n)$ on the polynomial algebra $\mathrm{Pol}(S_{n,m})$ consisting of complex-valued polynomial functions on $S_{n,m}$. Let $\mathrm{Pol}(S_{n,m})^K$ denote the subalgebra of $\mathrm{Pol}(S_{n,m})$ consisting of all polynomials on $S_{n,m}$ invariant under the representation σ of $O(n)$, and $\mathbb{D}(\mathcal{P}_{n,m})$ denote the algebra of all differential operators on $\mathcal{P}_{n,m}$ invariant under the action (1.2) of $GL_{n,m}$. We see that there is a canonically defined linear bijection of $\text{Pol}(S_{n,m})^K$ onto $\mathbb{D}(\mathcal{P}_{n,m})$ which is not multiplicative. We will see that $\mathbb{D}(\mathcal{P}_{n,m})$ is *not* commutative. The most important problem here is in finding a complete list of explicit generators of $\text{Pol}(S_{n,m})^K$ and a complete list of explicit generators of $\mathbb{D}(\mathcal{P}_{n,m})$. We propose several natural problems. We present some explicit invariant differential operators which may be useful. In Section 4, we deal with the case when $n = 1$. In Section 5, we deal with the case when $n = 2$ and $m = 1, 2$. In Section 6, we deal with the case when $n = 3$ and $m = 1, 2$. In Section 7, we deal with the case when $n = 4$ and $m = 1, 2$. In the final section, we present some open problems and discuss a notion of automorphic forms on $\mathcal{P}_{n,m}$ using $GL_{n,m}$-invariant differential operators on the Minkowski-Euclid space $\mathcal{P}_{n,m}$.

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Notations. Denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers, respectively. Denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers, respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\text{tr}(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transposed matrix of M. For a positive integer n, I_n denotes the identity matrix of degree n.

2. Review on invariant differential operators on $\mathcal{P}_n$

For a variable $Y = (y_{ij}) \in \mathcal{P}_n$, set

$$dY = (dy_{ij}) \quad \text{and} \quad \frac{\partial}{\partial Y} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial y_{ij}} \right),$$

where δ_{ij} denotes the Kronecker delta symbol.

For a fixed element $g \in GL(n, \mathbb{R})$, put

$$Y_* = g \cdot Y = gY\,{}^tg, \quad Y \in \mathcal{P}_n.$$

Then

$$dY_* = g\,dY\,{}^tg \quad \text{and} \quad \frac{\partial}{\partial Y_*} = {}^tg^{-1} \frac{\partial}{\partial Y}\, g^{-1}. \tag{2.1}$$

Consider the following differential operators

$$D_i = \text{tr}\left(\left(Y \frac{\partial}{\partial Y} \right)^i \right), \quad i = 1, 2, \ldots, n, \tag{2.2}$$

where $\mathrm{tr}(A)$ denotes the trace of a square matrix A. By Formula (2.1), we get

$$\left(Y_*\frac{\partial}{\partial Y_*}\right)^i = g\,\left(Y\frac{\partial}{\partial Y}\right)^i g^{-1}$$

for any $g \in GL(n,\mathbb{R})$. Hence each D_i is invariant under the action (1.1) of $GL(n,\mathbb{R})$.

Selberg [10] proved the following.

Theorem 2.1. *The algebra $\mathbb{D}(\mathcal{P}_n)$ of all differential operators on $\mathcal{P}_n$ invariant under the action* (1.1) *of $GL(n,\mathbb{R})$ is generated by $D_1, D_2, \ldots, D_n$. Furthermore, $D_1, D_2, \ldots, D_n$ are algebraically independent and $\mathbb{D}(\mathcal{P}_n)$ is isomorphic to the commutative ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$ with n indeterminates $x_1, x_2, \ldots, x_n$.*

Proof. The proof can be found in [4], p. 337, [8], pp. 64–66 and [11], pp. 29–30. The last statement follows immediately from the work of Harish-Chandra [1, 2] or [4], p. 294. □

Let $\mathfrak{g} = \mathbb{R}^{(n,n)}$ be the Lie algebra of $GL(n,\mathbb{R})$. The adjoint representation Ad of $GL(n,\mathbb{R})$ is given by

$$\mathrm{Ad}(g) = gXg^{-1}, \quad g \in GL(n,\mathbb{R}),\ X \in \mathfrak{g}.$$

The Killing form B of $\mathfrak{g}$ is given by

$$B(X,Y) = 2n\,\mathrm{tr}(XY) - 2\,\mathrm{tr}(X)\,\mathrm{tr}(Y), \quad X, Y \in \mathfrak{g}.$$

Since $B(aI_n, X) = 0$ for all $a \in \mathbb{R}$ and $X \in \mathfrak{g}$, B is degenerate. So the Lie algebra $\mathfrak{g}$ of $GL(n,\mathbb{R})$ is not semi-simple.

The Lie algebra $\mathfrak{k}$ of K is

$$\mathfrak{k} = \left\{\, X \in \mathfrak{g} \mid X + {}^tX = 0 \,\right\}.$$

Let $\mathfrak{p}$ be the subspace of $\mathfrak{g}$ defined by

$$\mathfrak{p} = \left\{\, X \in \mathfrak{g} \mid X = {}^tX \in \mathbb{R}^{(n,n)} \,\right\}.$$

Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is the direct sum of $\mathfrak{k}$ and $\mathfrak{p}$ with respect to the Killing form B. Since $\mathrm{Ad}(k)\mathfrak{p} \subset \mathfrak{p}$ for any $k \in K$, K acts on $\mathfrak{p}$ via the adjoint representation by

$$k \cdot X = \mathrm{Ad}(k)X = kX\,{}^tk, \quad k \in K,\ X \in \mathfrak{p}. \tag{2.3}$$

The action (2.3) induces the action of K on the polynomial algebra $\mathrm{Pol}(\mathfrak{p})$ of $\mathfrak{p}$ and the symmetric algebra $S(\mathfrak{p})$. Denote by $\mathrm{Pol}(\mathfrak{p})^K$ (resp., $S(\mathfrak{p})^K$) the subalgebra of $\mathrm{Pol}(\mathfrak{p})$ (resp., $S(\mathfrak{p})$) consisting of all K-invariants. The following inner product $(\ ,\ )$ on $\mathfrak{p}$ defined by

$$(X,Y) = B(X,Y), \quad X, Y \in \mathfrak{p}$$

gives an isomorphism as vector spaces

$$\mathfrak{p} \cong \mathfrak{p}^*, \quad X \mapsto f_X, \quad X \in \mathfrak{p}, \tag{2.4}$$

where $\mathfrak{p}^*$ denotes the dual space of $\mathfrak{p}$ and f_X is the linear functional on $\mathfrak{p}$ defined by

$$f_X(Y) = (Y, X), \quad Y \in \mathfrak{p}.$$

It is known that there is a canonical linear bijection of $S(\mathfrak{p})^K$ onto $\mathbb{D}(\mathcal{P}_n)$. Identifying $\mathfrak{p}$ with $\mathfrak{p}^*$ by the above isomorphism (2.4), we get a canonical linear bijection

$$\Theta_n : \mathrm{Pol}(\mathfrak{p})^K \longrightarrow \mathbb{D}(\mathcal{P}_n) \tag{2.5}$$

of $\mathrm{Pol}(\mathfrak{p})^K$ onto $\mathbb{D}(\mathcal{P}_n)$. The map Θ_n is described explicitly as follows. Put $N = n(n+1)/2$. Let $\{\xi_\alpha \mid 1 \le \alpha \le N\}$ be a basis of $\mathfrak{p}$. If $P \in \mathrm{Pol}(\mathfrak{p})^K$, then

$$\Big(\Theta_n(P)f\Big)(gK) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g \exp\left(\sum_{\alpha=1}^N t_\alpha \xi_\alpha\right) K\right)\right]_{(t_\alpha)=0}, \tag{2.6}$$

where $f \in C^\infty(\mathcal{P}_n)$. We refer the reader to [3, 4] for more detail. In general, it is difficult to express $\Theta_n(P)$ explicitly for a polynomial $P \in \mathrm{Pol}(\mathfrak{p})^K$.

Let

$$q_i(X) = \mathrm{tr}\big(X^i\big), \quad i = 1, 2, \ldots, n \tag{2.7}$$

be the polynomials on $\mathfrak{p}$. Here we take coordinates $x_{11}, x_{12}, \ldots, x_{nn}$ in $\mathfrak{p}$ given by

$$X = \begin{pmatrix} x_{11} & \frac{1}{2}x_{12} & \ldots & \frac{1}{2}x_{1n} \\ \frac{1}{2}x_{12} & x_{22} & \ldots & \frac{1}{2}x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}x_{1n} & \frac{1}{2}x_{2n} & \ldots & x_{nn} \end{pmatrix}.$$

For any $k \in K$,

$$(k \cdot q_i)(X) = q_i(k^{-1}Xk) = \mathrm{tr}(k^{-1}X^i k) = q_i(X), \quad i = 1, 2, \ldots, n.$$

Thus $q_i \in \mathrm{Pol}(\mathfrak{p})^K$ for $i = 1, 2, \ldots, n$. By a classical invariant theory (cf. [5, 12]), we can prove that the algebra $\mathrm{Pol}(\mathfrak{p})^K$ is generated by the polynomials $q_1, q_2, \ldots, q_n$ and that $q_1, q_2, \ldots, q_n$ are algebraically independent. Using Formula (2.6), we can show without difficulty that

$$\Theta_n(q_1) = \mathrm{tr}\left(2Y \frac{\partial}{\partial Y}\right).$$

However, $\Theta_n(q_i)$ $(i = 2, 3, \ldots, n)$ are yet known explicitly.

We propose the following conjecture.

Conjecture 1. For any n,

$$\Theta_n(q_i) = \operatorname{tr}\left(\left(2Y\frac{\partial}{\partial Y}\right)^i\right), \quad i = 1, 2, \ldots, n.$$

Remark. The author has verified that the above conjecture is true for $n = 1, 2$.

For a positive real number A,

$$ds^2_{n;A} = A \cdot \operatorname{tr}\bigl(Y^{-1}dY\, Y^{-1}dY\bigr)$$

is a Riemannian metric on $\mathcal{P}_n$ invariant under the action (1.1). The Laplacian $\Delta_{n;A}$ of $ds^2_{n;A}$ is given by

$$\Delta_{n;A} = \frac{1}{A}\operatorname{tr}\left(\left(Y\frac{\partial}{\partial Y}\right)^2\right).$$

For instance, consider the case when $n = 2$ and $A > 0$. If we write for $Y \in \mathcal{P}_2$,

$$Y = \begin{pmatrix} y_1 & y_3 \\ y_3 & y_2 \end{pmatrix} \quad \text{and} \quad \frac{\partial}{\partial Y} = \begin{pmatrix} \frac{\partial}{\partial y_1} & \frac{1}{2}\frac{\partial}{\partial y_3} \\ \frac{1}{2}\frac{\partial}{\partial y_3} & \frac{\partial}{\partial y_2} \end{pmatrix},$$

then

$$\begin{aligned} ds^2_{2;A} &= A\ \operatorname{tr}\bigl(Y^{-1}dY\, Y^{-1}dY\bigr) \\ &= \frac{A}{\left(y_1y_2 - y_3^2\right)^2}\ \Bigl\{y_2^2\, dy_1^2\ +\ y_1^2\, dy_2^2\ +\ 2\left(y_1y_2 + y_3^2\right) dy_3^2 \\ &\qquad +\ 2\, y_3^2\ dy_1dy_2 - 4\, y_2\, y_3\ dy_1dy_3 - 4\, y_1y_3\ dy_2dy_3\Bigr\} \end{aligned}$$

and its Laplacian $\Delta_{2;A}$ on $\mathcal{P}_2$ is

$$\begin{aligned} \Delta_{2;A} &= \frac{1}{A}\ \operatorname{tr}\left(\left(Y\frac{\partial}{\partial Y}\right)^2\right) \\ &= \frac{1}{A}\left\{y_1^2\,\frac{\partial^2}{\partial y_1^2} + y_2^2\,\frac{\partial^2}{\partial y_2^2} + \frac{1}{2}(y_1y_2 + y_3^2)\frac{\partial^2}{\partial y_3^2}\right. \\ &\qquad + 2\left(y_3^2\frac{\partial^2}{\partial y_1\partial y_2} + y_1y_3\frac{\partial^2}{\partial y_1\partial y_3} + y_2y_3\frac{\partial^2}{\partial y_2\partial y_3}\right) \\ &\qquad \left. + \frac{3}{2}\left(y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} + y_3\frac{\partial}{\partial y_3}\right)\right\}. \end{aligned}$$

3. Invariant differential operators on $\mathcal{P}_{n,m}$

For a variable $(Y, V) \in \mathcal{P}_{n,m}$ with $Y \in \mathcal{P}_n$ and $V \in \mathbb{R}^{(m,n)}$, put

$$Y = (y_{ij}) \text{ with } y_{ij} = y_{ji}, \quad V = (v_{kl}),$$
$$dY = (dy_{ij}), \quad dV = (dv_{kl}),$$

$$[dY] = \wedge_{i\le j} dy_{ij}, \qquad [dV] = \wedge_{k,l} dv_{kl},$$

and

$$\frac{\partial}{\partial Y} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial y_{ij}} \right), \quad \frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_{kl}} \right),$$

where $1 \le i, j, l \le n$ and $1 \le k \le m$.

For a fixed element $(g, \lambda) \in GL_{n,m}$, write

$$(Y_\star, V_\star) = (g, \lambda) \cdot (Y, V) = \big(g\, Y\, {}^t g, (V + \lambda)\, {}^t g\big),$$

where $(Y, V) \in \mathcal{P}_{n,m}$. Then we get

$$Y_\star = g\, Y\ {}^t g, \quad V_\star = (V + \lambda)\, {}^t g \tag{3.1}$$

and

$$\frac{\partial}{\partial Y_\star} = {}^t g^{-1} \frac{\partial}{\partial Y} g^{-1}, \quad \frac{\partial}{\partial V_\star} = \frac{\partial}{\partial V} g^{-1}. \tag{3.2}$$

Lemma 3.1. *For any two positive real numbers A and B, the following metric $ds^2_{n,m;A,B}$ on $\mathcal{P}_{n,m}$ defined by*

$$ds^2_{n,m;A,B} = A\, \sigma(Y^{-1} dY\, Y^{-1} dY) \,+\, B\, \sigma(Y^{-1}\, {}^t(dV)\, dV) \tag{3.3}$$

is a Riemannian metric on $\mathcal{P}_{n,m}$ which is invariant under the action (1.2) *of $GL_{n,m}$. The Laplacian $\Delta_{n,m;A,B}$ of $(\mathcal{P}_{n,m},\, ds^2_{n,m;A,B})$ is given by*

$$\Delta_{n,m;A,B} = \frac{1}{A}\, \sigma\left(\left(Y \frac{\partial}{\partial Y} \right)^2 \right) - \frac{m}{2\,A} \sigma \left(Y \frac{\partial}{\partial Y} \right) + \frac{1}{B} \sum_{k\le p} \left(\left(\frac{\partial}{\partial V} \right) Y\, {}^t\!\left(\frac{\partial}{\partial V} \right) \right)_{kp}.$$

Moreover, $\Delta_{n,m;A,B}$ is a differential operator of order 2 which is invariant under the action (1.2) *of $GL_{n,m}$.*

Proof. The proof can be found in [14]. □

Lemma 3.2. *The following volume element $dv_{n,m}(Y, V)$ on $\mathcal{P}_{n,m}$ defined by*

$$dv_{n,m}(Y, V) = (\det\, Y)^{-\frac{n+m+1}{2}} [dY][dV] \tag{3.4}$$

is invariant under the action (1.2) *of $GL_{n,m}$.*

Proof. The proof can be found in [14]. □

Theorem 3.1. *Any geodesic through the origin $(I_n, 0)$ for the Riemannian metric $ds^2_{n,m;1,1}$ is of the form*

$$\gamma(t) = \left(\lambda(2t)[k],\ Z \left(\int_0^t \lambda(t - s) ds \right) [k] \right),$$

where k is a fixed element of $O(n)$, Z is a fixed $h \times g$ real matrix, t is a real variable, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are fixed real numbers not all zero and

$$\lambda(t) := \mathrm{diag}\, (e^{\lambda_1 t}, \ldots, e^{\lambda_n t}).$$

Furthermore, the tangent vector $\gamma'(0)$ of the geodesic $\gamma(t)$ at $(I_n, 0)$ is $(D[k], Z)$, where $D = \mathrm{diag}\, (2\lambda_1, \ldots, 2\lambda_n)$.

Proof. The proof can be found in [14]. □

Theorem 3.2. *Let (Y_0,V_0) and (Y_1,V_1) be two points in $\mathcal{P}_{n,m}$. Let g be an element in $GL(n,\mathbb{R})$ such that $Y_0[\,{}^tg]=I_n$ and $Y_1[\,{}^tg]$ is diagonal. Then the length $s\big((Y_0,V_0),(Y_1,V_1)\big)$ of the geodesic joining (Y_0,V_0) and (Y_1,V_1) for the $GL_{n,m}$-invariant Riemannian metric $ds^2_{n,m;A,B}$ is given by*

(3.5)

$$s\big((Y_0,V_0),(Y_1,V_1)\big) = A\left\{\sum_{j=1}^n(\ln t_j)^2\right\}^{1/2} + B\int_0^1\left(\sum_{j=1}^n\Delta_j\, e^{-(\ln t_j)\,t}\right)^{1/2} dt,$$

where $\Delta_j=\sum_{k=1}^m \widetilde{v}_{kj}^2\ (1\le j\le n)$ with $(V_1-V_0)\,{}^tg=(\widetilde{v}_{kj})$ and $t_1,\dots,t_n$ denotes the zeros of $\det(t\,Y_0-Y_1)$.

Proof. The proof can be found in [14]. □

The Lie algebra $\mathfrak{g}_\star$ of $GL_{n,m}$ is given by

$$\mathfrak{g}_\star=\Big\{(X,Z)\mid X\in\mathbb{R}^{(n,n)},\ Z\in\mathbb{R}^{(m,n)}\Big\}$$

equipped with the following Lie bracket

$$\big[(X_1,Z_1),\,(X_2,Z_2)\big]=\big([X_1,X_2]_0,\,Z_2\,{}^tX_1-Z_1\,{}^tX_2\big),$$

where $[X_1,X_2]_0=X_1X_2-X_2X_1$ denotes the usual matrix bracket and (X_1,Z_1), $(X_2,Z_2)\in\mathfrak{g}_\star$. The adjoint representation $\mathrm{Ad}_\star$ of $GL_{n,m}$ is given by

$$\mathrm{Ad}_\star\,((g,\lambda))(X,Z)=\big(\,gXg^{-1},\,(Z-\lambda\,{}^tX)\,{}^tg\,\big), \tag{3.6}$$

where $(g,\lambda)\in GL_{n,m}$ and $(X,Z)\in\mathfrak{g}_\star$. Also, the adjoint representation $\mathrm{ad}_\star$ of $\mathfrak{g}_\star$ on $\mathrm{End}\,(\mathfrak{g}_\star)$ is given by

$$\mathrm{ad}_\star\,\big((X,Z)\big)\big((X_1,Z_1)\big)=\big[(X,Z),(X_1,Z_1)\big].$$

We see that the Killing form $B_\star$ of $\mathfrak{g}_\star$ is given by

$$B_\star\big((X_1,Z_1),(X_2,Z_2)\big)=(2\,n+m)\,\mathrm{tr}(X_1X_2)-2\,\mathrm{tr}(X_1)\,\mathrm{tr}(X_2).$$

The Lie algebra $\mathfrak{k}$ of K is

$$\mathfrak{k}=\Big\{(X,0)\in\mathfrak{g}_\star\ \Big|\ X+{}^tX=0\Big\}.$$

Let $\mathfrak{p}_\star$ be the subspace of $\mathfrak{g}_\star$ defined by

$$\mathfrak{p}_\star=\Big\{(X,Z)\in\mathfrak{g}_\star\ \Big|\ X={}^tX\in\mathbb{R}^{(n,n)},\ Z\in\mathbb{R}^{(m,n)}\Big\}.$$

Then we have the following relations

$$[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k}\qquad\text{and}\qquad[\mathfrak{k},\mathfrak{p}_\star]\subset\mathfrak{p}_\star.$$

In addition, we have

$$\mathfrak{g}_\star=\mathfrak{k}\oplus\mathfrak{p}_\star\qquad(\,\text{the direct sum}\,).$$

K acts on $\mathfrak{p}_\star$ via the adjoint representation $\mathrm{Ad}_\star$ of $GL_{n,m}$ by

$$k\cdot(X,Z)=\big(kX\,{}^tk,\,Z\,{}^tk\big),\quad k\in K,\ (X,Z)\in\mathfrak{p}_\star. \tag{3.7}$$

The action (3.7) induces the action of K on the polynomial algebra $\mathrm{Pol}(\mathfrak{p}_\star)$ of $\mathfrak{p}_\star$ and the symmetric algebra $S(\mathfrak{p}_\star)$. Denote by $\mathrm{Pol}(\mathfrak{p}_\star)^K$ (resp., $S(\mathfrak{p}_\star)^K$) the subalgebra of $\mathrm{Pol}(\mathfrak{p}_\star)$ (resp., $S(\mathfrak{p}_\star)$) consisting of all K-invariants. The following inner product $(\ ,\ )_\star$ on $\mathfrak{p}_\star$ defined by

$$\big((X_1,Z_1),(X_2,Z_2)\big)_\star=\mathrm{tr}(X_1X_2)+\mathrm{tr}(Z_1\,{}^tZ_2),\quad (X_1,Z_1),(X_2,Y_2)\in\mathfrak{p}_\star$$

gives an isomorphism as vector spaces

$$\mathfrak{p}_\star\cong\mathfrak{p}_\star^*,\quad (X,Z)\mapsto f_{X,Z},\quad (X,Z)\in\mathfrak{p}_\star, \tag{3.8}$$

where $\mathfrak{p}_\star^*$ denotes the dual space of $\mathfrak{p}_\star$ and $f_{X,Z}$ is the linear functional on $\mathfrak{p}_\star$ defined by

$$f_{X,Z}\big((X_1,Z_1)\big)=\big((X,Z),(X_1,Z_1)\big)_\star,\quad (X_1,Z_1)\in\mathfrak{p}_\star.$$

Let $\mathbb{D}(\mathcal{P}_{n,m})$ be the algebra of all differential operators on $\mathcal{P}_{n,m}$ that are invariant under the action (1.2) of $GL_{n,m}$. It is known that there is a canonical linear bijection of $S(\mathfrak{p}_\star)^K$ onto $\mathbb{D}(\mathcal{P}_{n,m})$. Identifying $\mathfrak{p}_\star$ with $\mathfrak{p}_\star^*$ by the above isomorphism (3.5), we get a canonical linear bijection

$$\Theta_{n,m}:\mathrm{Pol}(\mathfrak{p}_\star)^K\longrightarrow\mathbb{D}(\mathcal{P}_{n,m}) \tag{3.9}$$

of $\mathrm{Pol}(\mathfrak{p}_\star)^K$ onto $\mathbb{D}(\mathcal{P}_{n,m})$. The map $\Theta_{n,m}$ is described explicitly as follows. Put $N_\star=n(n+1)/2+mn$. Let $\{\eta_\alpha\,|\,1\le\alpha\le N_\star\}$ be a basis of $\mathfrak{p}_\star$. If $P\in\mathrm{Pol}(\mathfrak{p}_\star)^K$, then

$$\Big(\Theta_{n,m}(P)f\Big)(gK)=\left[P\left(\frac{\partial}{\partial t_\alpha}\right)f\left(g\exp\left(\sum_{\alpha=1}^{N_\star}t_\alpha\eta_\alpha\right)K\right)\right]_{(t_\alpha)=0}, \tag{3.10}$$

where $f\in C^\infty(\mathcal{P}_{n,m})$. We refer the reader to [4], pp. 280–289. In general, it is very hard to express $\Theta_{n,m}(P)$ explicitly for a polynomial $P\in\mathrm{Pol}(\mathfrak{p}_\star)^K$.

Take a coordinate (X,Z) in $\mathfrak{p}_\star$ such that

$$X=\begin{pmatrix} x_{11} & \frac12x_{12} & \dots & \frac12x_{1n}\\ \frac12x_{12} & x_{22} & \dots & \frac12x_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac12x_{1n} & \frac12x_{2n} & \dots & x_{nn}\end{pmatrix}\in\mathfrak{p}\quad\text{and}\quad Z=(z_{kl})\in\mathbb{R}^{(m,n)}.$$

Define the polynomials $\alpha_j,\ \beta_{pq}^{(k)},\ R_{jp}$ and S_{jp} on $\mathfrak{p}_\star$ by

$$\alpha_j(X,Z)=\mathrm{tr}\big(X^j\big),\quad 1\le j\le n, \tag{3.11}$$

$$\beta_{pq}^{(k)}(X,Z)=\big(Z\,X^k\,{}^tZ\big)_{pq},\quad 0\le k\le n-1,\ 1\le p\le q\le m, \tag{3.12}$$

$$R_{jp}(X,Z)=\mathrm{tr}\big(X^j({}^tZZ)^p\big),\quad 0\le j\le n-1,\ 1\le p\le m, \tag{3.13}$$

$$S_{jp}(X,Z)=\det\big(X^j({}^tZZ)^p\big),\quad 0\le j\le n-1,\ 1\le p\le m, \tag{3.14}$$

where $\big(Z^tZ\big)_{pq}$ $\big($resp.,$\big(ZX^tZ\big)_{pq}\big)$ denotes the (p,q)-entry of Z^tZ $\big($resp., $ZX^tZ\big)$.

For any $m\times m$ real matrix S, define the polynomials $M_{j;S}$, $Q_{p;S}$, $\Omega_{i,p,j;S}$ and $\Theta_{i,p,j;S}$ on $\mathfrak{p}_\star$ by

$$M_{j;S}(X,Z) = \operatorname{tr}\big((X+{}^tZSZ)^j\big), \quad 1\le j\le n, \tag{3.15}$$

$$Q_{p;S}(X,Z) = \operatorname{tr}\big(({}^tZ\,S\,Z)^p\big), \quad 1\le p\le n, \tag{3.16}$$

$$\Omega_{i,p,j;S}(X,Z) = \operatorname{tr}\Big(X^i({}^tZSZ)^p(X+{}^tZSZ)^j\Big), \tag{3.17}$$

$$\Theta_{i,p,j;S}(X,Z) = \det\Big(X^i({}^tZSZ)^p(X+{}^tZSZ)^j\Big), \tag{3.18}$$

where $0\le i,j\le n-1,\ 1\le p\le n$. We see that all α_j, $\beta_{pq}^{(k)}$, R_{jp}, S_{jp}, $M_{j;S}$, $Q_{p;S}$, $\Omega_{i,p,j;S}$ and $\Theta_{i,p,j;S}$ are elements of $\mathrm{Pol}(\mathfrak{p}_\star)^K$.

We propose the following natural problems.

Problem 1. Find a complete list of explicit generators of $\mathrm{Pol}(\mathfrak{p}_\star)^K$.

Problem 2. Find all relations among a set of generators of $\mathrm{Pol}(\mathfrak{p}_\star)^K$.

Problem 3. Find an easy or an effective way to express explicitly the images of the above invariant polynomials under the Helgason map $\Theta_{n,m}$.

Problem 4. Decompose $\mathrm{Pol}(\mathfrak{p}_\star)^K$ into $O(n)$-irreducibles.

Problem 5. Find a complete list of explicit generators of the algebra $\mathbb{D}(\mathcal{P}_{n,m})$ or construct explicit $GL_{n,m}$-invariant differential operators on $\mathcal{P}_{n,m}$.

Problem 6. Find all relations among a set of generators of $\mathbb{D}(\mathcal{P}_{n,m})$.

Problem 7. Is $\mathrm{Pol}(\mathfrak{p}_\star)^K$ finitely generated? Is $\mathbb{D}(\mathcal{P}_{n,m})$ finitely generated?

M. Itoh [6] proved the following theorem.

Theorem 3.3. *$Pol(\mathfrak{p}_\star)^K$ is generated by α_j $(1\le j\le n)$ and $\beta_{pq}^{(k)}$ $(0\le k\le n-1,\ 1\le p\le q\le m)$.*

Proof. We refer the reader to Theorem 3.1 in [6]. □

M. Itoh solved Problem 2 in [6], Theorem 3.2.

We present some invariant differential operators on $\mathcal{P}_{n,m}$. Define the differential operators D_j, Ω_{pq} and L_p on $\mathcal{P}_{n,m}$ by

$$D_j = \operatorname{tr}\left(\left(2Y\frac{\partial}{\partial Y}\right)^j\right), \quad 1\le j\le n, \tag{3.19}$$

(3.20)

$$\Omega_{pq}^{(k)} = \left\{\frac{\partial}{\partial V}\left(2Y\frac{\partial}{\partial Y}\right)^k Y\,{}^t\!\left(\frac{\partial}{\partial V}\right)\right\}_{pq}, \quad 0\le k\le n-1,\ 1\le p\le q\le m,$$

and

$$L_p = \operatorname{tr}\left(\left\{Y\,{}^t\!\left(\frac{\partial}{\partial V}\right)\frac{\partial}{\partial V}\right\}^p\right), \quad 1 \le p \le m. \tag{3.21}$$

Here, for a matrix A, we denote by A_{pq} the (p,q)-entry of A.

Also, define the differential operators S_{jp} by

$$S_{jp} = \operatorname{tr}\left(\left(2Y\frac{\partial}{\partial Y}\right)^j\left\{Y\,{}^t\!\left(\frac{\partial}{\partial V}\right)\frac{\partial}{\partial V}\right\}^p\right), \tag{3.22}$$

where $1 \le j \le n$ and $1 \le p \le m$.

For any real matrix S of degree m, define the differential operators $\Phi_{j;S}$, $L_{p;S}$ and $\Phi_{i,p,j;S}$ by

$$\Phi_{j;S} = \operatorname{tr}\left(\left\{Y\left(2\frac{\partial}{\partial Y} + {}^t\!\left(\frac{\partial}{\partial V}\right)S\left(\frac{\partial}{\partial V}\right)\right)\right\}^j\right), \quad 1 \le j \le n, \tag{3.23}$$

$$L_{p;S} = \operatorname{tr}\left(\left\{Y\,{}^t\!\left(\frac{\partial}{\partial V}\right)S\left(\frac{\partial}{\partial V}\right)\right\}^p\right), \quad 1 \le p \le m \tag{3.24}$$

and

(3.25)

$$\begin{aligned}&\Phi_{i,p,j;S}(X,Z)\\ &= \operatorname{tr}\left(\left(2Y\frac{\partial}{\partial Y}\right)^i\left(Y\,{}^t\!\left(\frac{\partial}{\partial V}\right)S\left(\frac{\partial}{\partial V}\right)\right)^p\left\{Y\left(2\frac{\partial}{\partial Y} + {}^t\!\left(\frac{\partial}{\partial V}\right)S\left(\frac{\partial}{\partial V}\right)\right)\right\}^j\right).\end{aligned}$$

We want to mention a special invariant differential operator on $\mathcal{P}_{n,m}$. In [13], the author studied the following differential operator $M_{n,m,\mathcal{M}}$ on $\mathcal{P}_{n,m}$ defined by

$$M_{n,m,\mathcal{M}} = \det(Y)\cdot\det\left(\frac{\partial}{\partial Y} + \frac{1}{8\pi}\,{}^t\!\left(\frac{\partial}{\partial V}\right)\mathcal{M}^{-1}\left(\frac{\partial}{\partial V}\right)\right), \tag{3.26}$$

where $\mathcal{M}$ is a positive definite, symmetric half-integral matrix of degree m. This differential operator characterizes *singular Jacobi forms*. For more detail, we refer the reader to [13]. From (3.1) and (3.2), we can easily see that the differential operator $M_{n,m,\mathcal{M}}$ is invariant under the action (1.2) of $GL_{n,m}$.

Question. Calculate the inverse of $M_{n,m,\mathcal{M}}$ under the Helgason map $\Theta_{n,m}$.

4. The case when $n = 1$

In this section, we consider the case when $n = m = 1$ and the case when $n = 1$ and $m \ge 2$ separately.

4.1. The case when $n = 1$ and $m = 1$

In this case,

$$GL_{1,1} = \mathbb{R}^{\times} \ltimes \mathbb{R}, \quad K = O(1), \quad \mathcal{P}_{1,1} = \mathbb{R}^{+} \times \mathbb{R},$$

where $\mathbb{R}^{\times} = \{a \in \mathbb{R} \mid a \neq 0\}$ and $\mathbb{R}^{+} = \{a \in \mathbb{R} \mid a > 0\}$. Clearly, $\mathfrak{k} = 0$ and $\mathfrak{p}_{\star} = \mathfrak{g}_{\star} = \{(x,z) \mid x, z \in \mathbb{R}\}$. Then $e = (1,0)$ and $f = (0,1)$ form the standard basis for $\mathfrak{p}_{\star}$. Using this basis, we take a coordinate (x,z) in $\mathfrak{p}_{\star}$; that is, if $w \in \mathfrak{p}_{\star}$, then we write $w = xe + zf$. We can show that $\mathrm{Pol}(\mathfrak{p}_{\star})^K$ is generated by the following polynomials

$$\alpha(x,z) = x \quad \text{and} \quad \beta(x,z) = z^2.$$

The generators α and β are *algebraically independent*. Let (y,v) be a coordinate in $\mathcal{P}_{1,1}$ with $y > 0$ and $v \in \mathbb{R}$. Then using Formula (3.10), we can show that

$$\Theta_{1,1}(\alpha) = 2y\,\frac{\partial}{\partial y} \quad \text{and} \quad \Theta_{1,1}(\beta) = y\,\frac{\partial^2}{\partial v^2}.$$

We see that $\Theta_{1,1}(\alpha)$ and $\Theta_{1,1}(\beta)$ generate the algebra $\mathbb{D}(\mathcal{P}_{1,1})$ and are *algebraically dependent*. Indeed, we have the following noncommutation relation

$$\Theta_{1,1}(\alpha)\Theta_{1,1}(\beta) - \Theta_{1,1}(\beta)\Theta_{1,1}(\alpha) = 2\,\Theta_{1,1}(\beta).$$

Hence the algebra $\mathbb{D}(\mathcal{P}_{1,1})$ is *not* commutative. The unitary dual $\widehat{K}$ of K consists of two elements. Let

$$\mathrm{Pol}(\mathfrak{p}_{\star}) = \sum_{\tau \in \widehat{K}} m_{\tau}\tau$$

be the decomposition of $\mathrm{Pol}(\mathfrak{p}_{\star})$ into K-irreducibles. It is easy to see that the multiplicity m_{τ} of τ is infinite for all $\tau \in \widehat{K}$. So the action of K on $\mathrm{Pol}(\mathfrak{p}_{\star})$ is not multiplicity-free. In this case, the seven problems proposed in Section 3 are completely solved.

4.2. The case when $n = 1$ and $m \geq 2$

Consider the case when $n = 1$ and $m \geq 2$. In this case,

$$GL_{1,m} = \mathbb{R}^{\times} \ltimes \mathbb{R}^{(m,1)}, \quad K = O(1), \quad \mathcal{P}_{1,m} = \mathbb{R}^{+} \times \mathbb{R}^{(m,1)},$$

where $\mathbb{R}^{\times} = \{a \in \mathbb{R} \mid a \neq 0\}$ and $\mathbb{R}^{+} = \{a \in \mathbb{R} \mid a > 0\}$. Clearly, $\mathfrak{k} = 0$ and $\mathfrak{p}_{\star} = \mathfrak{g}_{\star} = \left\{(x,z) \mid x \in \mathbb{R},\ z \in \mathbb{R}^{(m,1)}\right\}$. Let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbb{R}^{(m,1)}$. Then

$$\eta_0 = (1,0),\ \eta_1 = (0,e_1),\ \eta_2 = (0,e_2), \ldots,\ \eta_m = (0,e_m)$$

form a basis of $\mathfrak{p}_{\star}$. Using this basis, we take a coordinate $(x, z_1, z_2, \ldots, z_m)$ in $\mathfrak{p}_{\star}$; that is, if $w \in \mathfrak{p}_{\star}$, then we write $w = x\eta_0 + \sum_{k=1}^{m} z_k\eta_k$. We can show that $\mathrm{Pol}(\mathfrak{p}_{\star})^K$ is generated by the following polynomials

$$\alpha(x,z) = x \quad \text{and} \quad \beta_{kl}(x,z) = z_k\, z_l, \quad 1 \leq k \leq l \leq m,$$

where $z = (z_1, z_2, \ldots, z_m)$. We see easily that one has the following relations

$$\beta_{kk}\beta_{ll} = \beta_{kl}^2 \qquad \text{for } 1 \leq k < l \leq m$$

and

$$\beta_{kk}\beta_{ll}^2\beta_{pp} = \beta_{kl}^2\beta_{lp}^2 \qquad \text{for } 1 \le k < l < p \le m.$$

Therefore, the generators α and $\beta_{kl}\,(1 \le k \le l \le m)$ are *algebraically dependent*.

Let (y, v) be a coordinate in $\mathcal{P}_{1,m}$ with $y > 0$ and $v = {}^t(v_1, v_2, \ldots, v_m) \in \mathbb{R}^{(m,1)}$. Then using Formula (3.10), we can show that

$$\Theta_{1,m}(\alpha) = 2y\,\frac{\partial}{\partial y} \quad \text{and} \quad \Theta_{1,m}(\beta_{kl}) = y\frac{\partial^2}{\partial v_k \partial v_l}, \quad 1 \le k \le l \le m.$$

We see that $\Theta_{1,m}(\alpha)$ and $\Theta_{1,m}(\beta_{kl})\,(1 \le k \le l \le m)$ generate the algebra $\mathbb{D}(\mathcal{P}_{1,m})$. Although $\Theta_{1,m}(\beta_{kl})\,(1 \le k \le l \le m)$ commute with each other, $\Theta_{1,m}(\alpha)$ does not commute with any $\Theta_{1,m}(\beta_{kl})$. Indeed, we have the noncommutation relation

$$\Theta_{1,m}(\alpha)\Theta_{1,m}(\beta_{kl}) - \Theta_{1,m}(\beta_{kl})\Theta_{1,m}(\alpha) = 2\,\Theta_{1,m}(\beta_{kl}).$$

Hence the algebra $\mathbb{D}(\mathcal{P}_{1,m})$ is *not* commutative. It is easily seen that the action of K on $\mathrm{Pol}(\mathfrak{p}_\star)$ is *not* multiplicity-free.

5. The case when $n = 2$

In this section, we deal with the case when $n = 2,\ m = 1$ and the case when $n = m = 2$.

5.1. The case when $n = 2$ and $m = 1$

In this case,

$$GL_{2,1} = GL(2, \mathbb{R}) \ltimes \mathbb{R}^{(1,2)}, \quad K = O(2) \ \text{ and } \ GL_{2,1}/K = \mathcal{P}_2 \times \mathbb{R}^{(1,2)} = \mathcal{P}_{2,1}.$$

We see easily that

$$\mathfrak{p}_\star = \left\{ (X, Z) \mid X = {}^tX \in \mathbb{R}^{(2,2)}, \quad Z \in \mathbb{R}^{(1,2)} \right\}.$$

Put

$$e_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right), \quad e_2 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right), \quad e_3 = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0 \right)$$

and

$$f_1 = (0, (1, 0)), \qquad f_2 = (0, (0, 1)).$$

Then $\{\, e_1,\, e_2,\, e_3,\, f_1,\, f_2 \,\}$ forms a basis for $\mathfrak{p}_\star$. For variables $(X, Z) \in \mathfrak{p}_\star$, write

$$X = \begin{pmatrix} x_1 & \frac{1}{2}x_3 \\ \frac{1}{2}x_3 & x_2 \end{pmatrix} \quad \text{and} \quad Z = (z_1, z_2).$$

The following polynomials

$$\alpha_1(X, Z) = \mathrm{tr}(X) = x_1 + x_2, \qquad \alpha_2(X, Z) = \mathrm{tr}(X^2) = x_1^2 + x_2^2 + \frac{1}{2}\,x_3^2,$$

$$\xi(X, Z) = Z\,{}^tZ = z_1^2 + z_2^2$$

and

$$\varphi(X,Z) = ZX\,{}^tZ \,=\, x_1\, z_1^2 \,+\, x_2\, z_2^2 \,+\, x_3\, z_1 z_2$$

generate the algebra $\text{Pol}(\mathfrak{p}_\star)^K$. We can show that the invariants α_1, α_2, ξ and φ are *algebraically independent*. We omit the detail.

Now we compute the $GL_{2,1}$-invariant differential operators D_1, D_2, Ψ, Δ on $\mathcal{P}_{2,1}$ corresponding to the K-invariants α_1, α_2, ξ, φ, respectively, under a canonical linear bijection

$$\Theta_{2,1}\,:\, \text{Pol}(\mathfrak{p}_\star)^K \longrightarrow \mathbb{D}(\mathcal{P}_{2,1}).$$

For real variables $t=(t_1,t_2,t_3)$ and $s=(s_1,s_2)$, we have

$$\begin{aligned} &\exp\left(t_1e_1+t_2e_2+t_3e_3+s_1f_1+s_2f_2\right)\\ &= \left(\begin{pmatrix} a_1(t,s) & a_3(t,s)\\ a_3(t,s) & a_2(t,s)\end{pmatrix},\, (\,b_1(t,s),\, b_2(t,s)\,)\right),\end{aligned}$$

where

$$\begin{aligned} a_1(t,s) &= 1+t_1+\frac{1}{2!}(t_1^2+t_3^2)+\frac{1}{3!}(t_1^3+2t_1t_3^2+t_2t_3^2)+\cdots\,,\\ a_2(t,s) &= 1+t_2+\frac{1}{2!}(t_2^2+t_3^2)+\frac{1}{3!}(t_1t_3^2+2t_2t_3^2+t_2^3)+\cdots\,,\\ a_3(t,s) &= t_3+\frac{1}{2!}(t_1+t_2)t_3+\frac{1}{3!}(t_1t_2+t_1^2+t_2^2+t_3^2)t_3+\cdots\,,\\ b_1(t,s) &= s_1-\frac{1}{2!}(s_1t_1+s_2t_3)+\frac{1}{3!}\left\{\,s_1(t_1^2+t_3^2)+s_2(t_1t_3+t_2t_3)\right\}-\cdots\,,\\ b_2(t,s) &= s_2-\frac{1}{2!}(s_1t_3+s_2t_2)+\frac{1}{3!}\left\{\,s_1(t_1+t_2)t_3+s_2(t_2^2+t_3^2)\right\}-\cdots\,.\end{aligned}$$

For brevity, we write a_i, b_k for $a_i(t,s)$, $b_k(t,s)$ $(\,i=1,2,3,\ k=1,2\,)$, respectively. We now fix an element $(g,c)\in GL_{2,1}$ and write

$$g=\begin{pmatrix} g_1 & g_{12}\\ g_{21} & g_2\end{pmatrix} \quad \text{and} \quad c=(c_1,c_2).$$

Put

$$\big(Y(t,s),\,V(t,s)\big)=\left((g,c)\cdot\exp\left(\sum_{i=1}^{3}t_ie_i+\sum_{k=1}^{2}s_kf_k\right)\right)\cdot(I_2,0)$$

with

$$Y(t,s)=\begin{pmatrix} y_1(t,s) & y_3(t,s)\\ y_3(t,s) & y_2(t,s)\end{pmatrix} \quad \text{and} \quad V(t,s)=(v_1(t,s),\,v_2(t,s)).$$

By an easy computation, we obtain

$$\begin{aligned} y_1 &= (g_1a_1+g_{12}a_3)^2+(g_1a_3+g_{12}a_2)^2,\\ y_2 &= (g_{21}a_1+g_2a_3)^2+(g_{21}a_3+g_2a_2)^2,\\ y_3 &= (g_1a_1+g_{12}a_3)(g_{21}a_1+g_2a_3)+(g_1a_3+g_{12}a_2)(g_{21}a_3+g_2a_2),\end{aligned}$$

$$v_1 = (c_1 + b_1a_1 + b_2a_3)g_1 + (c_2 + b_1a_3 + b_2a_2)g_{12},$$
$$v_2 = (c_1 + b_1a_1 + b_2a_3)g_{21} + (c_2 + b_1a_3 + b_2a_2)g_2.$$

Using the chain rule, we can easily compute the $GL_{2,1}$-invariant differential operators $D_1 = \Theta_{2,1}(\alpha_1)$, $D_2 = \Theta_{2,1}(\alpha_2)$, $\Psi = \Theta_{2,1}(\xi)$ and $\Delta = \Theta_{2,1}(\varphi)$. They are given by

$$\begin{aligned}
D_1 &= 2\,\mathrm{tr}\left(Y\frac{\partial}{\partial Y}\right) = 2\left(y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} + y_3\frac{\partial}{\partial y_3}\right),\\
D_2 &= \mathrm{tr}\left(\left(2Y\frac{\partial}{\partial Y}\right)^2\right)\\
&= 3\,D_1 \,+\, 8\left(y_3^2\,\frac{\partial^2}{\partial y_1\partial y_2} \,+\, y_1y_3\,\frac{\partial^2}{\partial y_1\partial y_3} \,+\, y_2y_3\,\frac{\partial^2}{\partial y_2\partial y_3}\right)\\
&\quad +\, 4\left\{y_1^2\,\frac{\partial^2}{\partial y_1^2} \,+\, y_2^2\,\frac{\partial^2}{\partial y_2^2} \,+\, \frac{1}{2}(y_1y_2 + y_3^2)\,\frac{\partial^2}{\partial y_3^2}\right\},\\
\Psi &= \mathrm{tr}\left(Y\,{}^t\!\left(\frac{\partial}{\partial V}\right)\left(\frac{\partial}{\partial V}\right)\right)\\
&= y_1\,\frac{\partial^2}{\partial v_1^2} \,+\, 2y_3\,\frac{\partial^2}{\partial v_1\partial v_2} \,+\, y_2\,\frac{\partial^2}{\partial v_2^2}
\end{aligned}$$

and

$$\begin{aligned}
\Delta &= \frac{\partial}{\partial V}\left(2Y\frac{\partial}{\partial Y}\right)Y\,{}^t\!\left(\frac{\partial}{\partial V}\right)\\
&= 2\left(y_1^2\,\frac{\partial^3}{\partial y_1\partial v_1^2} \,+\, 2\,y_1y_3\,\frac{\partial^3}{\partial y_1\partial v_1\partial v_2} \,+\, y_3^2\,\frac{\partial^3}{\partial y_1\partial v_2^2}\right)\\
&\quad +\, 2\left(y_3^2\,\frac{\partial^3}{\partial y_2\partial v_1^2} \,+\, 2\,y_2\,y_3\,\frac{\partial^3}{\partial y_2\partial v_1\partial v_2} \,+\, y_2^2\,\frac{\partial^3}{\partial y_2\partial v_2^2}\right)\\
&\quad +\, 2\left\{y_1y_3\,\frac{\partial^3}{\partial y_3\partial v_1^2} \,+\, \left(y_1y_2 \,+\, y_3^2\right)\frac{\partial^3}{\partial y_3\partial v_1\partial v_2} \,+\, y_2y_3\,\frac{\partial^3}{\partial y_3\partial v_2^2}\right\}\\
&\quad +\, 3\left(y_1\,\frac{\partial^2}{\partial v_1^2} \,+\, 2y_3\,\frac{\partial^2}{\partial v_1\partial v_2} \,+\, y_2\,\frac{\partial^2}{\partial v_2^2}\right).
\end{aligned}$$

Clearly, D_1 commutes with D_2 but Ψ does not commute with D_1 nor with D_2. Indeed, we have the following noncommutation relations

$$[D_1,\,\Psi] = D_1\Psi - \Psi D_1 = 2\,\Psi$$

and

$$\begin{aligned}
[D_2,\,\Psi] &= D_2\Psi - \Psi D_2\\
&= 2\,(\,2\,D_1\,-1\,)\,\Psi -\, 8\det(Y)\cdot\det\left(\frac{\partial}{\partial Y} + {}^t\!\left(\frac{\partial}{\partial V}\right)\frac{\partial}{\partial V}\right)
\end{aligned}$$

$$+\ 8\det(Y)\cdot\det\left(\frac{\partial}{\partial Y}\right)\ -\ 4\left(y_1y_2+y_3^2\right)\frac{\partial^3}{\partial y_3\partial v_1\partial v_2}.$$

Hence the algebra $\mathbb{D}(\mathcal{P}_{2,1})$ is *not* commutative.

5.2. The case when $n = 2$ and $m = 2$

In this case,

$$GL_{2,2} = GL(2,\mathbb{R}) \ltimes \mathbb{R}^{(2,2)},\quad K = O(2)\ \text{ and }\ GL_{2,2}/K = \mathcal{P}_2 \times \mathbb{R}^{(2,2)} = \mathcal{P}_{2,2}.$$

We see easily that

$$\mathfrak{p}_\star = \left\{(X,Z)\mid X = {}^tX \in \mathbb{R}^{(2,2)},\quad Z \in \mathbb{R}^{(2,2)}\right\}.$$

Let O_2 be the 2×2 zero matrix. Put

$$e_1 = \left(\begin{pmatrix}1&0\\0&0\end{pmatrix}, O_2\right),\quad e_2 = \left(\begin{pmatrix}0&0\\0&1\end{pmatrix}, O_2\right),\quad e_3 = \left(\begin{pmatrix}0&\frac12\\\frac12&0\end{pmatrix}, O_2\right)$$

and

$$f_1 = \left(O_2, \begin{pmatrix}1&0\\0&0\end{pmatrix}\right),\quad f_2 = \left(O_2, \begin{pmatrix}0&1\\0&0\end{pmatrix}\right),$$
$$f_3 = \left(O_2, \begin{pmatrix}0&0\\1&0\end{pmatrix}\right), f_4 = \left(O_2, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right).$$

Then $\{\,e_1,\,e_2,\,e_3,\,f_1,\,f_2,\,f_3,\,f_4\,\}$ forms a basis for $\mathfrak{p}_\star$. For variables $(X,Z)\in\mathfrak{p}_\star$, write

$$X = \begin{pmatrix}x_1 & \frac12 x_3\\ \frac12 x_3 & x_2\end{pmatrix}\quad\text{and}\quad Z = \begin{pmatrix}z_{11}&z_{12}\\z_{21}&z_{22}\end{pmatrix}.$$

From Theorem 3.3, the algebra $\mathrm{Pol}(\mathfrak{p}_\star)^K$ is generated by the following polynomials

$$\alpha_1(X,Z) = \mathrm{tr}(X) = x_1 + x_2,$$
$$\alpha_2(X,Z) = \mathrm{tr}(X^2) = x_1^2 + x_2^2 + \frac12 x_3^2,$$
$$\beta_{11}^{(0)}(X,Z) = (Z\,{}^tZ)_{11} = z_{11}^2 + z_{12}^2,$$
$$\beta_{12}^{(0)}(X,Z) = (Z\,{}^tZ)_{12} = z_{11}z_{21} + z_{12}z_{22},$$
$$\beta_{22}^{(0)}(X,Z) = (Z\,{}^tZ)_{22} = z_{21}^2 + z_{22}^2,$$
$$\beta_{11}^{(1)}(X,Z) = (ZX\,{}^tZ)_{11} = x_1z_{11}^2 + x_2z_{12}^2 + x_3z_{11}z_{12},$$
$$\beta_{12}^{(1)}(X,Z) = (ZX\,{}^tZ)_{12} = x_1z_{11}z_{21} + x_2z_{12}z_{22} + \frac12 x_3(z_{11}z_{22} + z_{12}z_{21}),$$
$$\beta_{22}^{(1)}(X,Z) = (ZX\,{}^tZ)_{22} = x_1z_{21}^2 + x_2z_{22}^2 + x_3z_{21}z_{22}.$$

Set

$$\Delta_{ab} := \det \begin{pmatrix} \beta_{11}^{(a)} & \beta_{12}^{(b)} \\ \beta_{12}^{(a)} & \beta_{22}^{(b)} \end{pmatrix} \qquad \text{for } a, b = 0, 1.$$

By a direct computation, we can show that the following equation

$$\alpha_1 \Delta_{00} - \Delta_{01} - \Delta_{10} = 0 \tag{5.1}$$

holds.

We take a coordinate (Y, V) in $\mathcal{P}_{2,2}$, that is,

$$Y = \begin{pmatrix} y_1 & y_3 \\ y_3 & y_2 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}.$$

Put

$$\frac{\partial}{\partial Y} = \begin{pmatrix} \frac{\partial}{\partial y_1} & \frac{1}{2}\frac{\partial}{\partial y_3} \\ \frac{1}{2}\frac{\partial}{\partial y_3} & \frac{\partial}{\partial y_2} \end{pmatrix} \quad \text{and} \quad \frac{\partial}{\partial V} = \begin{pmatrix} \frac{\partial}{\partial v_{11}} & \frac{\partial}{\partial v_{12}} \\ \frac{\partial}{\partial v_{21}} & \frac{\partial}{\partial v_{22}} \end{pmatrix}.$$

Consider the following differential operators

$$D_i := \operatorname{tr}\left(\left(2\, Y \frac{\partial}{\partial Y} \right)^i \right), \quad i = 1, 2$$

and

$$\Omega_{pq}^{(k)} = \left\{ \frac{\partial}{\partial V} \left(2Y \frac{\partial}{\partial Y} \right)^k Y \, {}^t\!\left(\frac{\partial}{\partial V} \right) \right\}_{pq}, \quad k = 0, 1, \ 1 \le p \le q \le 2.$$

Note that D_1, D_2, $\Omega_{11}^{(0)}, \ldots, \Omega_{22}^{(1)}$ are $GL_{2,2}$-invariant. For brevity, we put

$$\partial_{ij} = \frac{\partial}{\partial v_{ij}}, \quad i, j = 1, 2.$$

It is easily seen that

$$\begin{aligned}
D_1 &= \operatorname{tr}\left(2\, Y \frac{\partial}{\partial Y} \right) = 2 \sum_{i=1}^{3} y_i \frac{\partial}{\partial y_i}, \\
D_2 &= 3\, D_1 + 8 \left(y_3^2 \frac{\partial^2}{\partial y_1 \partial y_2} + y_1 y_3 \frac{\partial^2}{\partial y_1 \partial y_3} + y_2 y_3 \frac{\partial^2}{\partial y_2 \partial y_3} \right) \\
&\quad + 4 \left\{ y_1^2 \frac{\partial^2}{\partial y_1^2} + y_2^2 \frac{\partial^2}{\partial y_2^2} + \frac{1}{2}\big(y_1 y_2 + y_3^2\big) \frac{\partial^2}{\partial y_3^2} \right\}, \\
\Omega_{11}^{(0)} &= y_1\, \partial_{11}^2 + y_2\, \partial_{12}^2 + 2\, y_3\, \partial_{11}\partial_{12}, \\
\Omega_{12}^{(0)} &= y_1\, \partial_{11}\partial_{21} + y_2\, \partial_{12}\partial_{22} + y_3\, (\partial_{11}\partial_{22} + \partial_{12}\partial_{21}), \\
\Omega_{22}^{(0)} &= y_1\, \partial_{21}^2 + y_2\, \partial_{22}^2 + 2\, y_3\, \partial_{21}\partial_{22}.
\end{aligned}$$

Then by a direct computation, we have the following relations

$$[D_1, D_2] = 0, \tag{5.2}$$

$$[\Omega^{(0)}_{kl},\Omega^{(0)}_{pq}]=0,\quad 1\le k\le l\le 2,\ 1\le p\le q\le 2, \tag{5.3}$$

$$[D_1,\Omega^{(0)}_{11}]=2\,\Omega^{(0)}_{11},\quad [D_1,\Omega^{(0)}_{12}]=2\,\Omega^{(0)}_{12},\quad [D_1,\Omega^{(0)}_{22}]=2\,\Omega^{(0)}_{22}. \tag{5.4}$$

Therefore, $\mathbb{D}(\mathcal{P}_{2,2})$ is not commutative.

6. The case when $n=3$

6.1. The case when $n=3$ and $m=1$

In this case,

$$GL_{3,1}=GL(3,\mathbb{R})\ltimes\mathbb{R}^{(1,3)},\quad K=O(3)\ \text{ and }\ GL_{3,1}/K=\mathcal{P}_3\times\mathbb{R}^{(1,3)}=\mathcal{P}_{3,1}.$$

We see easily that

$$\mathfrak{p}_\star=\left\{(X,Z)\mid X={}^tX\in\mathbb{R}^{(3,3)},\quad Z\in\mathbb{R}^{(1,3)}\right\}.$$

Put

$$E_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad E_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad E_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},$$

$$E_4=\begin{pmatrix}0&\frac12&0\\\frac12&0&0\\0&0&0\end{pmatrix},\quad E_5=\begin{pmatrix}0&0&\frac12\\0&0&0\\\frac12&0&0\end{pmatrix},\quad E_6=\begin{pmatrix}0&0&0\\0&0&\frac12\\0&\frac12&0\end{pmatrix}.$$

Let O_3 be the 3×3 zero matrix and let $O_{1,3}=(0,0,0)\in\mathbb{R}^{(1,3)}$. Put

$$e_i=(E_i,O_{1,3}),\quad 1\le i\le 6,$$
$$f_1=(O_3,(1,0,0)),\quad f_2=(O_3,(0,1,0)),\quad f_3=(O_3,(0,0,1)).$$

Then $\{\,e_i,\ f_j\,|\,1\le i\le 6,\ 1\le j\le 3\,\}$ forms a basis for $\mathfrak{p}_\star$. Using this basis, we write for variables $(X,Z)\in\mathfrak{p}_\star$,

$$X=\begin{pmatrix}x_1&\frac12x_4&\frac12x_5\\\frac12x_4&x_2&\frac12x_6\\\frac12x_5&\frac12x_6&x_3\end{pmatrix}\quad\text{and}\quad Z=(z_1,z_2,z_3).$$

From Theorem 3.3, the algebra $\mathrm{Pol}(\mathfrak{p}_\star)^K$ is generated by the following polynomials

$$\alpha_1(X,Z)=x_1+x_2+x_3,$$

$$\alpha_2(X,Z)=x_1^2+x_2^2+x_3^2+\frac12\big(x_4^2+x_5^2+x_6^2\big),$$

$$\alpha_3(X,Z)=x_1^3+x_2^3+x_3^3+\frac34\ \big\{(x_1+x_2)x_4^2+(x_1+x_3)x_5^2+(x_2+x_3)x_6^2\big\}$$
$$+\ \frac34\,x_4x_5x_6,$$

$$\beta_0(X,Z)=z_1^2+z_2^2+z_3^2,$$

$$\beta_1(X,Z) = x_1 z_1^2 + x_2 z_2^2 + x_3 z_3^2 + x_4 z_1 z_2 + x_5 z_1 z_3 + x_6 z_2 z_3,$$

$$\begin{aligned}\beta_2(X,Z) = {} & x_1^2 z_1^2 + x_2^2 z_2^2 + \frac{1}{4}\,\left\{ \left(x_4^2+x_5^2\right) z_1^2 \,+\, \left(x_4^2+x_6^2\right) z_2^2 \,+\, \left(x_5^2+x_6^2\right) z_3^2 \right\} \\ & + \left(x_1x_4 + x_2x_4 + \frac{1}{2} x_5 x_6 \right) z_1 z_2 \,+\, \left(x_1x_5 + x_3x_5 + \frac{1}{2} x_4 x_6 \right) z_1 z_3 \\ & + \left(x_2x_6 + x_3x_6 + \frac{1}{2} x_4 x_5 \right) z_2 z_3.\end{aligned}$$

We take a coordinate (Y,V) in $\mathcal{P}_{3,1}$, that is,

$$Y = \begin{pmatrix} y_1 & y_4 & y_5 \\ y_4 & y_2 & y_6 \\ y_5 & y_6 & y_3 \end{pmatrix} \quad \text{and} \quad V = (v_1, v_2, v_3).$$

Put

$$\frac{\partial}{\partial Y} = \begin{pmatrix} \frac{\partial}{\partial y_1} & \frac{1}{2}\frac{\partial}{\partial y_4} & \frac{1}{2}\frac{\partial}{\partial y_5} \\ \frac{1}{2}\frac{\partial}{\partial y_4} & \frac{\partial}{\partial y_2} & \frac{1}{2}\frac{\partial}{\partial y_6} \\ \frac{1}{2}\frac{\partial}{\partial y_5} & \frac{1}{2}\frac{\partial}{\partial y_6} & \frac{\partial}{\partial y_3} \end{pmatrix} \quad \text{and} \quad \frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}, \frac{\partial}{\partial v_3} \right).$$

Consider the following differential operators

$$D_i := \operatorname{tr}\left(\left(2\,Y \frac{\partial}{\partial Y} \right)^i \right), \quad i = 1,2,3$$

and

$$\Omega_k = \frac{\partial}{\partial V} \left(2Y \frac{\partial}{\partial Y} \right)^k Y \, {}^t\!\left(\frac{\partial}{\partial V} \right), \quad k = 0,1,2.$$

Note that D_1, D_2, D_3, Ω_0, Ω_1 and Ω_2 are $GL_{2,2}$-invariant. It is easily seen that

$$D_1 = \operatorname{tr}\left(2\,Y \frac{\partial}{\partial Y} \right) = 2 \sum_{i=1}^{6} y_i \frac{\partial}{\partial y_i},$$

$$\begin{aligned}\Omega_0 = {} & y_1 \, \frac{\partial^2}{\partial v_1^2} + y_2 \, \frac{\partial^2}{\partial v_2^2} + y_3 \, \frac{\partial^2}{\partial v_3^2} \\ & + \, 2\, y_4 \, \frac{\partial^2}{\partial v_1 \partial v_2} \,+\, 2\, y_5 \, \frac{\partial^2}{\partial v_1 \partial v_3} \,+\, 2\, y_6 \, \frac{\partial^2}{\partial v_2 \partial v_3}.\end{aligned}$$

Then we have the following relations

(6.1) $$[D_i, D_j] = 0 \quad \text{for all } i,j = 1,2,3$$

and

(6.2) $$[D_1, \Omega_0] = 2\,\Omega_0.$$

Therefore, $\mathbb{D}(\mathcal{P}_{3,1})$ is not commutative.

6.2. The case when $n = 3$ and $m = 2$

In this case,

$$GL_{3,2} = GL(3,\mathbb{R}) \ltimes \mathbb{R}^{(2,3)}, \quad K = O(3) \ \text{ and } \ GL_{3,2}/K = \mathcal{P}_3 \times \mathbb{R}^{(2,3)} = \mathcal{P}_{3,2}.$$

We see easily that

$$\mathfrak{p}_\star = \left\{ (X,Z) \mid X = {}^tX \in \mathbb{R}^{(3,3)}, \quad Z \in \mathbb{R}^{(2,3)} \right\}.$$

Put

$$\begin{aligned} E_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ E_4 &= \begin{pmatrix} 0 & \frac12 & 0 \\ \frac12 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_5 = \begin{pmatrix} 0 & 0 & \frac12 \\ 0 & 0 & 0 \\ \frac12 & 0 & 0 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac12 \\ 0 & \frac12 & 0 \end{pmatrix}. \end{aligned}$$

and

$$\begin{aligned} F_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ F_4 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Let O_3 be the 3×3 zero matrix and let

$$O_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{(2,3)}.$$

Put

$$e_i = (E_i, O_{2,3}), \quad f_j = (O_3, F_j) \quad 1 \le i,j \le 6.$$

Then $\{\, e_i,\, f_j \,|\, 1 \le i,j \le 6 \,\}$ forms a basis for $\mathfrak{p}_\star$. Using this basis, we write for variables $(X,Z) \in \mathfrak{p}_\star$,

$$X = \begin{pmatrix} x_1 & \frac12 x_4 & \frac12 x_5 \\ \frac12 x_4 & x_2 & \frac12 x_6 \\ \frac12 x_5 & \frac12 x_6 & x_3 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \end{pmatrix}.$$

From Theorem 3.3, the algebra $\mathrm{Pol}(\mathfrak{p}_\star)^K$ is generated by the following polynomials

$$\begin{aligned} \alpha_1(X,Z) &= x_1 + x_2 + x_3, \\ \alpha_2(X,Z) &= x_1^2 + x_2^2 + x_3^2 + \frac12\big(x_4^2 + x_5^2 + x_6^2\big), \\ \alpha_3(X,Z) &= x_1^3 + x_2^3 + x_3^3 + \frac34\, \big\{(x_1+x_2)x_4^2 + (x_1+x_3)x_5^2 + (x_2+x_3)x_6^2\big\} \\ &\quad + \frac34\, x_4x_5x_6, \\ \beta_{11}^{(0)}(X,Z) &= z_{11}^2 + z_{12}^2 + z_{13}^2, \\ \beta_{12}^{(0)}(X,Z) &= z_{11}z_{21} + z_{12}z_{22} + z_{13}z_{23}, \end{aligned}$$

$$\beta_{22}^{(0)}(X,Z) = z_{21}^2 + z_{22}^2 + z_{23}^2,$$
$$\beta_{11}^{(1)}(X,Z) = x_1 z_{11}^2 + x_2 z_{12}^2 + x_3 z_{13}^2 + x_4 z_{11} z_{12} + x_5 z_{11} z_{13} + x_6 z_{12} z_{13},$$
$$\begin{aligned}\beta_{12}^{(1)}(X,Z) = {} & x_1 z_{11} z_{21} + x_2 z_{12} z_{22} + x_3 z_{13} z_{23} + \frac{1}{2}\, x_4 (z_{11} z_{22} + z_{12} z_{21}) \\ & + \frac{1}{2}\, x_5 (z_{11} z_{23} + z_{13} z_{21}) + \frac{1}{2}\, x_6 (z_{12} z_{23} + z_{13} z_{22}),\end{aligned}$$
$$\beta_{22}^{(1)}(X,Z) = x_1 z_{21}^2 + x_2 z_{22}^2 + x_3 z_{23}^2 + x_4 z_{21} z_{22} + x_5 z_{21} z_{23} + x_6 z_{22} z_{23},$$
$$\begin{aligned}\beta_{11}^{(2)}(X,Z) = {} & x_1^2 z_{11}^2 + x_2^2 z_{12}^2 + x_3^2 z_{13}^2 \\ & + \frac{1}{4}\, \left\{ x_4^2 \big(z_{11}^2 + z_{12}^2\big) + x_5^2 \big(z_{11}^2 + z_{13}^2\big) + x_6^2 \big(z_{12}^2 + z_{13}^2\big) \right\} \\ & + (x_1 + x_2)\, x_4 z_{11} z_{12} + (x_1 + x_3)\, x_5 z_{11} z_{13} + (x_2 + x_3)\, x_6 z_{12} z_{13} \\ & + \frac{1}{2}\, \big(x_4 x_5 z_{12} z_{13} + x_4 x_6 z_{11} z_{13} + x_5 x_6 z_{11} z_{12}\big),\end{aligned}$$
$$\begin{aligned}\beta_{12}^{(2)}(X,Z) = {} & x_1^2 z_{11} z_{21} + x_2^2 z_{12} z_{22} + x_3^2 z_{13} z_{23} \\ & + \frac{1}{4}\, \left\{ \big(x_4^2 + x_5^2\big)\, z_{11} z_{21} + \big(x_4^2 + x_6^2\big)\, z_{12} z_{22} + \big(x_5^2 + x_6^2\big)\, z_{13} z_{23} \right\} \\ & + \frac{1}{2} \left(x_1 x_4 + x_2 x_4 + \frac{1}{2} x_5 x_6 \right) (z_{11} z_{22} + z_{12} z_{21}) \\ & + \frac{1}{2} \left(x_1 x_5 + x_3 x_5 + \frac{1}{2} x_4 x_6 \right) (z_{11} z_{23} + z_{13} z_{21}) \\ & + \frac{1}{2} \left(x_2 x_6 + x_3 x_6 + \frac{1}{2} x_4 x_5 \right) (z_{12} z_{23} + z_{13} z_{22}),\end{aligned}$$
$$\begin{aligned}\beta_{22}^{(2)}(X,Z) = {} & x_1^2 z_{21}^2 + x_2^2 z_{22}^2 + x_3^2 z_{23}^2 \\ & + \frac{1}{4}\, \left\{ x_4^2 \big(z_{21}^2 + z_{22}^2\big) + x_5^2 \big(z_{21}^2 + z_{23}^2\big) + x_6^2 \big(z_{22}^2 + z_{23}^2\big) \right\} \\ & + (x_1 + x_2)\, x_4 z_{21} z_{22} + (x_1 + x_3)\, x_5 z_{21} z_{23} + (x_2 + x_3)\, x_6 z_{22} z_{23} \\ & + \frac{1}{2}\, \big(x_4 x_5 z_{22} z_{23} + x_4 x_6 z_{21} z_{23} + x_5 x_6 z_{21} z_{22}\big).\end{aligned}$$

Set

$$\Delta_{ab} := \det \begin{pmatrix} \beta_{11}^{(a)} & \beta_{12}^{(b)} \\ \beta_{12}^{(a)} & \beta_{22}^{(b)} \end{pmatrix} \qquad \text{for } a, b = 0, 1, 2.$$

By a direct computation, we can show that

$$\big(\alpha_1^2 - \alpha_2\big)\, \Delta_{00} - 2\, \alpha_1 \big(\Delta_{01} + \Delta_{10}\big) + 2 \big(\Delta_{02} + \Delta_{11} + \Delta_{20}\big) = 0. \tag{6.3}$$

We take a coordinate (Y, V) in $\mathcal{P}_{3,2}$, that is,

$$Y = \begin{pmatrix} y_1 & y_4 & y_5 \\ y_4 & y_2 & y_6 \\ y_5 & y_6 & y_3 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \end{pmatrix}.$$

Put

$$\frac{\partial}{\partial Y} = \begin{pmatrix} \frac{\partial}{\partial y_1} & \frac{1}{2}\frac{\partial}{\partial y_4} & \frac{1}{2}\frac{\partial}{\partial y_5} \\ \frac{1}{2}\frac{\partial}{\partial y_4} & \frac{\partial}{\partial y_2} & \frac{1}{2}\frac{\partial}{\partial y_6} \\ \frac{1}{2}\frac{\partial}{\partial y_5} & \frac{1}{2}\frac{\partial}{\partial y_6} & \frac{\partial}{\partial y_3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{\partial}{\partial v_{11}} & \frac{\partial}{\partial v_{12}} & \frac{\partial}{\partial v_{13}} \\ \frac{\partial}{\partial v_{21}} & \frac{\partial}{\partial v_{22}} & \frac{\partial}{\partial v_{23}} \end{pmatrix}.$$

Consider the following differential operators

$$D_i := \operatorname{tr}\left(\left(2\,Y\frac{\partial}{\partial Y}\right)^i\right), \quad i = 1, 2, 3$$

and

$$\Omega_{pq}^{(k)} = \left\{ \frac{\partial}{\partial V}\left(2Y\frac{\partial}{\partial Y}\right)^k Y\, {}^t\!\left(\frac{\partial}{\partial V}\right)\right\}_{pq}, \quad k = 0, 1, 2,\ 1 \le p \le q \le 2.$$

Note that D_1, D_2, D_3, $\Omega_{11}^{(0)}, \ldots, \Omega_{22}^{(2)}$ are $GL_{3,2}$-invariant. For brevity, we put

$$\partial_{ij} = \frac{\partial}{\partial v_{ij}}, \quad i = 1, 2,\ j = 1, 2, 3.$$

It is easily seen that

$$\begin{aligned} D_1 &= \operatorname{tr}\left(2\,Y\frac{\partial}{\partial Y}\right) = 2\sum_{i=1}^{6} y_i \frac{\partial}{\partial y_i}, \\ \Omega_{11}^{(0)} &= y_1\,\partial_{11}^2 + y_2\,\partial_{12}^2 + y_3\,\partial_{13}^2 + 2\,y_4\,\partial_{11}\partial_{12} + 2\,y_5\,\partial_{11}\partial_{13} + 2\,y_6\,\partial_{12}\partial_{13}, \\ \Omega_{12}^{(0)} &= y_1\,\partial_{11}\partial_{21} + y_2\,\partial_{12}\partial_{22} + y_3\,\partial_{13}\partial_{23} + y_4\left(\partial_{11}\partial_{22} + \partial_{12}\partial_{21}\right) \\ &\quad + y_5\left(\partial_{11}\partial_{23} + \partial_{13}\partial_{21}\right) + y_6\left(\partial_{12}\partial_{23} + \partial_{13}\partial_{22}\right), \\ \Omega_{22}^{(0)} &= y_1\,\partial_{21}^2 + y_2\,\partial_{22}^2 + y_3\,\partial_{23}^2 + 2\,y_4\,\partial_{21}\partial_{22} + 2\,y_5\,\partial_{21}\partial_{23} + 2\,y_6\,\partial_{22}\partial_{23}. \end{aligned}$$

Then we have the following relations

$$[D_i, D_j] = 0 \quad \text{for all } i, j = 1, 2, 3, \tag{6.4}$$

$$[\Omega_{kl}^{(0)}, \Omega_{pq}^{(0)}] = 0, \quad 1 \le k \le l \le 2,\ 1 \le p \le q \le 2 \tag{6.5}$$

and

$$[D_1, \Omega_{11}^{(0)}] = 2\,\Omega_{11}^{(0)}, \quad [D_1, \Omega_{12}^{(0)}] = 2\,\Omega_{12}^{(0)}, \quad [D_1, \Omega_{22}^{(0)}] = 2\,\Omega_{22}^{(0)}. \tag{6.6}$$

Therefore, $\mathbb{D}(\mathcal{P}_{3,2})$ is not commutative.

7. The case when $n = 4$

6.1. The case when $n = 4$ and $m = 1$

In this case,

$$GL_{4,1} = GL(4, \mathbb{R}) \ltimes \mathbb{R}^{(1,4)}, \quad K = O(4) \text{ and } GL_{4,1}/K = \mathcal{P}_4 \times \mathbb{R}^{(1,4)} = \mathcal{P}_{4,1}.$$

We see easily that

$$\mathfrak{p}_\star = \left\{ (X,Z) \mid X = {}^tX \in \mathbb{R}^{(4,4)}, \quad Z \in \mathbb{R}^{(1,4)} \right\}.$$

Put

$$E_1 = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},\ E_2 = \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},\ E_3 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix},\ E_5 = \begin{pmatrix} 0&\frac12&0&0\\\frac12&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},\ E_6 = \begin{pmatrix} 0&0&\frac12&0\\0&0&0&0\\\frac12&0&0&0\\0&0&0&0 \end{pmatrix},$$

$$E_7 = \begin{pmatrix} 0&0&0&\frac12\\0&0&0&0\\0&0&0&0\\\frac12&0&0&0 \end{pmatrix},\ E_8 = \begin{pmatrix} 0&0&0&0\\0&0&\frac12&0\\0&\frac12&0&0\\0&0&0&0 \end{pmatrix},$$

$$E_9 = \begin{pmatrix} 0&0&0&0\\0&0&0&\frac12\\0&0&0&0\\0&\frac12&0&0 \end{pmatrix},\ E_{10} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&\frac12\\0&0&\frac12&0 \end{pmatrix}.$$

Let O_4 be the 4×4 zero matrix and let $O_{1,4} = (0,0,0,0) \in \mathbb{R}^{(1,4)}$. Put

$$\begin{aligned} &e_i = (E_i, O_{1,4}), \quad 1 \le i \le 10,\\ &f_1 = (O_4,(1,0,0,0)),\ f_2 = (O_4,(0,1,0,0)),\\ &f_3 = (O_4,(0,0,1,0)),\ f_4 = (O_4,(0,0,0,1)). \end{aligned}$$

Then $\{\, e_i,\ f_j \,|\, 1 \le i \le 10,\ 1 \le j \le 4 \}$ forms a basis for $\mathfrak{p}_\star$. Using this basis, we write for variables $(X,Z) \in \mathfrak{p}_\star$,

$$X = \begin{pmatrix} x_1 & \frac12 x_5 & \frac12 x_6 & \frac12 x_7\\ \frac12 x_5 & x_2 & \frac12 x_8 & \frac12 x_9\\ \frac12 x_6 & \frac12 x_8 & x_3 & \frac12 x_{10}\\ \frac12 x_7 & \frac12 x_9 & \frac12 x_{10} & x_4 \end{pmatrix} \quad \text{and} \quad Z = (z_1, z_2, z_3, z_4).$$

Put

$$A = x_1^2 + \frac14\, x_5^2 + \frac14\, x_6 + \frac14\, x_7^2, \tag{7.1}$$

$$B = x_2^2 + \frac14\, x_5^2 + \frac14\, x_8 + \frac14\, x_9^2, \tag{7.2}$$

$$C = x_3^2 + \frac14\, x_6^2 + \frac14\, x_8 + \frac14\, x_{10}^2, \tag{7.3}$$

$$D = x_4^2 + \frac14\, x_7^2 + \frac14\, x_9 + \frac14\, x_{10}^2, \tag{7.4}$$

$$E = \frac{1}{2}(x_1+x_2)\,x_5 + \frac{1}{4}(x_6\,x_8 + x_7\,x_9), \tag{7.5}$$

$$F = \frac{1}{2}(x_1+x_3)\,x_6 + \frac{1}{4}(x_3\,x_6 + x_5\,x_8), \tag{7.6}$$

$$G = \frac{1}{2}(x_1+x_4)\,x_7 + \frac{1}{4}(x_5\,x_9 + x_6\,x_{10}), \tag{7.7}$$

$$H = \frac{1}{2}(x_2+x_3)\,x_8 + \frac{1}{4}(x_5\,x_6 + x_9\,x_{10}), \tag{7.8}$$

$$I = \frac{1}{2}(x_2+x_4)\,x_9 + \frac{1}{4}(x_5\,x_7 + x_8\,x_{10}), \tag{7.9}$$

$$J = \frac{1}{2}(x_3+x_4)\,x_{10} + \frac{1}{4}(x_6\,x_{10} + x_6\,x_7). \tag{7.10}$$

From Theorem 3.3, the algebra $\mathrm{Pol}(\mathfrak{p}_\star)^K$ is generated by the following polynomials

$$\begin{aligned}
\alpha_1(X,Z) &= x_1+x_2+x_3+x_4,\\
\alpha_2(X,Z) &= x_1^2+x_2^2+x_3^2+x_4^2+\frac{1}{2}\big(x_5^2+x_6^2+x_7^2+x_8^2+x_9^2+x_{10}^2\big),\\
\alpha_3(X,Z) &= x_1^3+x_2^3+x_3^3+x_4^3\\
&\quad+\frac{3}{4}\,x_1\big(x_5^2+x_6^2+x_7^2\big)+\frac{3}{4}\,x_2\big(x_5^2+x_8^2+x_9^2\big)\\
&\quad+\frac{3}{4}\,x_3\big(x_6^2+x_8^2+x_{10}^2\big)+\frac{3}{4}\,x_4\big(x_7^2+x_9^2+x_{10}^2\big)\\
&\quad+\frac{3}{4}\big(x_5\,x_6\,x_8+x_5\,x_7\,x_9+x_6\,x_7\,x_{10}+x_8\,x_9\,x_{10}\big),\\
\alpha_4(X,Z) &= A^2+B^2+C^2+D^2+2\big(E^2+F^2+G^2+H^2+I^2+J^2\big),\\
\beta_0(X,Z) &= z_1^2+z_2^2+z_3^2+z_4^2,\\
\beta_1(X,Z) &= x_1z_1^2+x_2z_2^2+x_3z_3^2+x_4z_4^2,\\
&\quad+x_5z_1z_2+x_6z_1z_3+x_7z_1z_4+x_8z_2z_3+x_9z_2z_4+x_{10}z_3z_4,\\
\beta_2(X,Z) &= Az_1^2+Bz_2^2+Cz_3^2+Dz_4^2,\\
&\quad+2\big(Ez_1z_2+Fz_1z_3+Gz_1z_4+Hz_2z_3+Iz_2z_4+Jz_3z_4\big),\\
\beta_3(X,Z) &= \frac{1}{2}\big(2Ax_1+Ex_5+Fx_6+Gx_7\big)z_1^2\\
&\quad+\frac{1}{2}\big(2Bx_2+Ex_5+Hx_8+Ix_9\big)z_2^2\\
&\quad+\frac{1}{2}\big(2Cx_3+Fx_6+Hx_8+Jx_{10}\big)z_3^2\\
&\quad+\frac{1}{2}\big(2Dx_4+Gx_7+Ix_9+Jx_{10}\big)z_4^2\\
&\quad+\frac{1}{2}\left\{2E(x_1+x_2)+(A+B)x_5+Hx_6+Ix_7+Fx_8+Gx_9\right\}z_1z_2
\end{aligned}$$

$$+\frac{1}{2}\left\{2F(x_1+x_3)+Hx_5+(A+C)x_6+Jx_7+Ex_8+Gx_{10}\right\}z_1z_3$$
$$+\frac{1}{2}\left\{2G(x_1+x_4)+Ix_5+Jx_6+(A+D)x_7+Ex_9+Fx_{10}\right\}z_1z_4$$
$$+\frac{1}{2}\left\{2H(x_2+x_3)+Fx_5+Ex_6+(B+C)x_8+Jx_9+Ix_{10}\right\}z_2z_3$$
$$+\frac{1}{2}\left\{2I(x_2+x_4)+Gx_5+Ex_7+Jx_8+(B+D)x_9+Hx_{10}\right\}z_2z_4$$
$$+\frac{1}{2}\left\{2J(x_3+x_4)+Gx_6+Fx_7+Ix_8+Hx_9+(C+D)x_{10}\right\}z_3z_4.$$

We take a coordinate (Y,V) in $\mathcal{P}_{4,1}$, that is,

$$Y=\begin{pmatrix} y_1 & y_5 & y_6 & y_7\\ y_5 & y_2 & y_8 & y_9\\ y_6 & y_8 & y_3 & y_{10}\\ y_7 & y_9 & y_{10} & y_4\end{pmatrix}\quad\text{and}\quad V=(v_1,v_2,v_3,v_4).$$

Put

$$\frac{\partial}{\partial Y}=\begin{pmatrix} \frac{\partial}{\partial y_1} & \frac12\frac{\partial}{\partial y_5} & \frac12\frac{\partial}{\partial y_6} & \frac12\frac{\partial}{\partial y_7}\\ \frac12\frac{\partial}{\partial y_5} & \frac{\partial}{\partial y_2} & \frac12\frac{\partial}{\partial y_8} & \frac12\frac{\partial}{\partial y_9}\\ \frac12\frac{\partial}{\partial y_6} & \frac12\frac{\partial}{\partial y_8} & \frac{\partial}{\partial y_3} & \frac12\frac{\partial}{\partial y_{10}}\\ \frac12\frac{\partial}{\partial y_7} & \frac12\frac{\partial}{\partial y_9} & \frac12\frac{\partial}{\partial y_{10}} & \frac{\partial}{\partial y_4}\end{pmatrix}\quad\text{and}\quad \frac{\partial}{\partial V}=\left(\frac{\partial}{\partial v_1},\frac{\partial}{\partial v_2},\frac{\partial}{\partial v_3},\frac{\partial}{\partial v_4}\right).$$

Let

$$D_i=\operatorname{tr}\left(\left(2\,Y\frac{\partial}{\partial Y}\right)^i\right),\quad i=1,2,3,4$$

and

$$\Omega_j=\frac{\partial}{\partial V}\left(2\,Y\frac{\partial}{\partial Y}\right)^j\,Y\,{}^t\!\left(\frac{\partial}{\partial V}\right),\quad j=0,1,2,3.$$

It is easily seen that

$$D_1=\operatorname{tr}\left(2\,Y\frac{\partial}{\partial Y}\right)=2\sum_{i=1}^{10}y_i\frac{\partial}{\partial y_i}.$$

For brevity, we put

$$\partial_i=\frac{\partial}{\partial v_i},\quad i=1,2,3,4.$$

Then we get

$$\begin{aligned}\Omega_0=\;&y_1\,\partial_1^2+y_2\,\partial_2^2+y_3\,\partial_3^2+y_4\,\partial_4^2+2\,y_5\,\partial_1\partial_2\\ &+2\,y_6\,\partial_1\partial_3+2\,y_7\,\partial_1\partial_4+2\,y_8\,\partial_2\partial_3+2\,y_9\,\partial_2\partial_4+2\,y_{10}\,\partial_3\partial_4.\end{aligned}$$

We observe that D_1, D_2, D_3, D_4, Ω_0, $\Omega_1,\Omega_2,\Omega_3$ are invariant differential operators in $\mathbb{D}(\mathcal{P}_{4,1})$. Then we have the following relations

$$[D_i,D_j]=0\quad\text{for all } i,j=1,2,3,4 \tag{7.11}$$

and

$$[D_1, \Omega_0] = 2\,\Omega_0. \tag{7.12}$$

Therefore, $\mathbb{D}(\mathcal{P}_{4,1})$ is not commutative.

6.2. The case when $n = 4$ and $m = 2$

In this case,

$GL_{4,2} = GL(4,\mathbb{R}) \ltimes \mathbb{R}^{(2,4)}$, $K = O(4)$ and $\mathcal{P}_{4,2} = GL_{4,2}/K = \mathcal{P}_4 \times \mathbb{R}^{(2,4)}$.

We see easily that

$$\mathfrak{p}_\star = \left\{ (X, Z) \mid X = {}^tX \in \mathbb{R}^{(4,4)},\quad Z \in \mathbb{R}^{(2,4)} \right\}.$$

Put

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ E_5 = \begin{pmatrix} 0 & \frac12 & 0 & 0 \\ \frac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ E_6 = \begin{pmatrix} 0 & 0 & \frac12 & 0 \\ 0 & 0 & 0 & 0 \\ \frac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_7 = \begin{pmatrix} 0 & 0 & 0 & \frac12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac12 & 0 & 0 & 0 \end{pmatrix},\ E_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac12 & 0 \\ 0 & \frac12 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_9 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac12 \\ 0 & 0 & 0 & 0 \\ 0 & \frac12 & 0 & 0 \end{pmatrix},\ E_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac12 \\ 0 & 0 & \frac12 & 0 \end{pmatrix}.$$

Let O_4 be the 4×4 zero matrix and let

$$O_{2,4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{(2,4)}.$$

Put

$$e_i = (E_i, O_{2,4}),\quad 1 \le i \le 10,$$

$$f_1 = \left(O_4, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right),\ f_2 = \left(O_4, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right),$$

$$f_3 = \left(O_4, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right),\ f_4 = \left(O_4, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right),$$

$$f_5 = \left(O_4, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right),\ f_6 = \left(O_4, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right),$$

$$f_7 = \left(O_4, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right), \ f_8 = \left(O_4, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right).$$

Then $\{\, e_i,\, f_j \,|\, 1 \le i \le 10,\ 1 \le j \le 8 \,\}$ forms a basis for $\mathfrak{p}_\star$. Using this basis, we write for variables $(X,Z) \in \mathfrak{p}_\star$,

$$X = \begin{pmatrix} x_1 & \frac{1}{2}x_5 & \frac{1}{2}x_6 & \frac{1}{2}x_7 \\ \frac{1}{2}x_5 & x_2 & \frac{1}{2}x_8 & \frac{1}{2}x_9 \\ \frac{1}{2}x_6 & \frac{1}{2}x_8 & x_3 & \frac{1}{2}x_{10} \\ \frac{1}{2}x_7 & \frac{1}{2}x_9 & \frac{1}{2}x_{10} & x_4 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \end{pmatrix}.$$

Set

$$\Box_{11} = \frac{1}{2}\big(2A\,x_1 + E\,x_5 + F\,x_6 + G\,x_7\big),$$
$$\Box_{22} = \frac{1}{2}\big(2B\,x_2 + E\,x_5 + H\,x_8 + I\,x_9\big),$$
$$\Box_{33} = \frac{1}{2}\big(2C\,x_3 + F\,x_6 + H\,x_8 + J\,x_{10}\big),$$
$$\Box_{44} = \frac{1}{2}\big(2D\,x_4 + G\,x_7 + I\,x_9 + J\,x_{10}\big),$$
$$\Box_{12} = \frac{1}{2}\left\{2E(x_1+x_2) + (A+B)x_5 + Hx_6 + Ix_7 + Fx_8 + Gx_9\right\},$$
$$\Box_{13} = \frac{1}{2}\left\{2F(x_1+x_3) + Hx_5 + (A+C)x_6 + Jx_7 + Ex_8 + Gx_{10}\right\},$$
$$\Box_{14} = \frac{1}{2}\left\{2G(x_1+x_4) + Ix_5 + Jx_6 + (A+D)x_7 + Ex_9 + Fx_{10}\right\},$$
$$\Box_{23} = \frac{1}{2}\left\{2H(x_2+x_3) + Fx_5 + Ex_6 + (B+C)x_8 + Jx_9 + Ix_{10}\right\},$$
$$\Box_{24} = \frac{1}{2}\left\{2I(x_2+x_4) + Gx_5 + Ex_7 + Jx_8 + (B+D)x_9 + Hx_{10}\right\},$$
$$\Box_{34} = \frac{1}{2}\left\{2J(x_3+x_4) + Gx_6 + Fx_7 + Ix_8 + Hx_9 + (C+D)x_{10}\right\}.$$

From Theorem 3.3, the algebra $\mathrm{Pol}(\mathfrak{p}_\star)^K$ is generated by the following 16 polynomials

$$\alpha_1(X,Z) = x_1 + x_2 + x_3 + x_4,$$
$$\alpha_2(X,Z) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + \frac{1}{2}\big(x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2 + x_{10}^2\big),$$
$$\begin{aligned}\alpha_3(X,Z) = {} & x_1^3 + x_2^3 + x_3^3 + x_4^3 \\ & + \frac{3}{4}\,x_1\big(x_5^2 + x_6^2 + x_7^2\big) + \frac{3}{4}\,x_2\big(x_5^2 + x_8^2 + x_9^2\big) \\ & + \frac{3}{4}\,x_3\big(x_6^2 + x_8^2 + x_{10}^2\big) + \frac{3}{4}\,x_4\big(x_7^2 + x_9^2 + x_{10}^2\big) \\ & + \frac{3}{4}\big(x_5\,x_6\,x_8 + x_5\,x_7\,x_9 + x_6\,x_7\,x_{10} + x_8\,x_9\,x_{10}\big),\end{aligned}$$

$$
\begin{aligned}
\alpha_4(X,Z) &= A^2+B^2+C^2+D^2+2\left(E^2+F^2+G^2+H^2+I^2+J^2\right),\\
\beta_{11}^{(0)}(X,Z) &= z_{11}^2+z_{12}^2+z_{13}^2+z_{14}^2,\\
\beta_{12}^{(0)}(X,Z) &= z_{11}z_{21}+z_{12}z_{22}+z_{13}z_{23}+z_{14}z_{24},\\
\beta_{22}^{(0)}(X,Z) &= z_{21}^2+z_{22}^2+z_{23}^2+z_{24}^2,\\
\beta_{11}^{(1)}(X,Z) &= x_1z_{11}^2+x_2z_{12}^2+x_3z_{13}^2+x_4z_{14}^2+x_5z_{11}z_{12}\\
&\quad + x_6z_{11}z_{13}+x_7z_{11}z_{14}+x_8z_{12}z_{13}+x_9z_{12}z_{14}+x_{10}z_{13}z_{14},\\
\beta_{12}^{(1)}(X,Z) &= x_1z_{11}z_{21}+x_2z_{12}z_{22}+x_3z_{13}z_{23}+x_4z_{14}z_{24}\\
&\quad + \frac{1}{2}\,x_5\,(z_{11}z_{22}+z_{12}z_{21})+\frac{1}{2}\,x_6\,(z_{11}z_{23}+z_{13}z_{21})\\
&\quad + \frac{1}{2}\,x_7\,(z_{11}z_{24}+z_{14}z_{21})+\frac{1}{2}\,x_8\,(z_{12}z_{23}+z_{13}z_{22})\\
&\quad + \frac{1}{2}\,x_9\,(z_{12}z_{24}+z_{14}z_{22})+\frac{1}{2}\,x_{10}\,(z_{13}z_{24}+z_{14}z_{23}),\\
\beta_{22}^{(1)}(X,Z) &= x_1z_{21}^2+x_2z_{22}^2+x_3z_{23}^2+x_4z_{24}^2+x_5z_{21}z_{22}\\
&\quad + x_6z_{21}z_{23}+x_7z_{21}z_{23}++x_8z_{22}z_{23}+x_9z_{22}z_{24}+x_{10}z_{23}z_{24},\\
\beta_{11}^{(2)}(X,Z) &= A\,z_{11}^2+B\,z_{12}^2+C\,z_{13}^2+D\,z_{14}^2+2\,E\,z_{11}z_{12}+2\,F\,z_{11}z_{13}\\
&\quad + 2\,G\,z_{11}z_{14}+2\,H\,z_{12}z_{13}+2\,I\,z_{12}z_{14}+2\,J\,z_{13}z_{14},\\
\beta_{12}^{(2)}(X,Z) &= A\,z_{11}z_{21}+B\,z_{12}z_{22}+C\,z_{13}z_{23}+D\,z_{14}z_{24}\\
&\quad + E\left(z_{11}z_{22}+z_{12}z_{21}\right)+F\left(z_{11}z_{23}+z_{13}z_{21}\right)\\
&\quad + G\left(z_{11}z_{24}+z_{14}z_{21}\right)+H\left(z_{12}z_{23}+z_{13}z_{22}\right)\\
&\quad + I\left(z_{12}z_{24}+z_{14}z_{22}\right)+J\left(z_{13}z_{24}+z_{14}z_{23}\right),\\
\beta_{22}^{(2)}(X,Z) &= A\,z_{21}^2+B\,z_{22}^2+C\,z_{23}^2+D\,z_{24}^2+2\,E\,z_{21}z_{22}+2\,F\,z_{21}z_{23}\\
&\quad + 2\,G\,z_{21}z_{24}+2\,H\,z_{22}z_{23}+2\,I\,z_{22}z_{24}+2\,J\,z_{23}z_{24},\\
\beta_{11}^{(3)}(X,Z) &= \Box_{11}\,z_{11}^2+\Box_{22}\,z_{12}^2+\Box_{33}\,z_{13}^2+\Box_{44}\,z_{14}^2+\Box_{12}\,z_{11}z_{12}\\
&\quad + \Box_{13}\,z_{11}z_{13}+\Box_{14}\,z_{11}z_{14}+\Box_{23}\,z_{12}z_{13}\\
&\quad + \Box_{24}\,z_{12}z_{14}+\Box_{34}\,z_{13}z_{14},\\
\beta_{12}^{(3)}(X,Z) &= \Box_{11}\,z_{11}z_{21}+\Box_{22}\,z_{12}z_{22}+\Box_{33}\,z_{13}z_{23}+\Box_{44}\,z_{14}z_{24}\\
&\quad + \Box_{12}\,z_{11}z_{22}+\Box_{13}\,z_{11}z_{23}+\Box_{14}\,z_{11}z_{24}+\Box_{23}\,z_{12}z_{23}\\
&\quad + \Box_{24}\,z_{12}z_{24}+\Box_{34}\,z_{13}z_{24},\\
\beta_{22}^{(3)}(X,Z) &= \Box_{11}\,z_{21}^2+\Box_{22}\,z_{22}^2+\Box_{33}\,z_{23}^2+\Box_{44}\,z_{24}^2+\Box_{12}\,z_{21}z_{22}\\
&\quad + \Box_{13}\,z_{21}z_{23}+\Box_{14}\,z_{21}z_{24}+\Box_{23}\,z_{22}z_{23}\\
&\quad + \Box_{24}\,z_{22}z_{24}+\Box_{34}\,z_{23}z_{24}.
\end{aligned}
$$

Here, $A, B, C, \ldots, J$ are defined as in (7.1)-(7.10).

Set

$$\Delta_{ab} := \det \begin{pmatrix} \beta_{11}^{(a)} & \beta_{12}^{(b)} \\ \beta_{12}^{(a)} & \beta_{22}^{(b)} \end{pmatrix} \qquad \text{for } a, b = 0, 1, 2, 3.$$

By a tedious direct computation, we can show that

$$\begin{aligned} (7.13) \qquad & \left(\alpha_1^3 - 3\,\alpha_1\alpha_2 \,+\, 2\,\alpha_3\right) \Delta_{00} \,-\, 3\left(\alpha_1^2 - \alpha_2\right)\left(\Delta_{01} \,+\, \Delta_{10}\right) \\ & + 6\,\alpha_1\left(\Delta_{02} \,+\, \Delta_{11} \,+\, \Delta_{20}\right) \,+\, 6\left(\Delta_{03} \,+\, \Delta_{12} \,+\, \Delta_{21} \,+\, \Delta_{30}\right) = 0. \end{aligned}$$

Take a coordinate (Y, V) in $\mathcal{P}_{4,2}$, that is,

$$Y = \begin{pmatrix} y_1 & y_5 & y_6 & y_7 \\ y_5 & y_2 & y_8 & y_9 \\ y_6 & y_8 & y_3 & y_{10} \\ y_7 & y_9 & y_{10} & y_4 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \end{pmatrix}.$$

Put

$$\frac{\partial}{\partial Y} = \begin{pmatrix} \frac{\partial}{\partial y_1} & \frac{1}{2}\frac{\partial}{\partial y_5} & \frac{1}{2}\frac{\partial}{\partial y_6} & \frac{1}{2}\frac{\partial}{\partial y_7} \\ \frac{1}{2}\frac{\partial}{\partial y_5} & \frac{\partial}{\partial y_2} & \frac{1}{2}\frac{\partial}{\partial y_8} & \frac{1}{2}\frac{\partial}{\partial y_9} \\ \frac{1}{2}\frac{\partial}{\partial y_6} & \frac{1}{2}\frac{\partial}{\partial y_8} & \frac{\partial}{\partial y_3} & \frac{1}{2}\frac{\partial}{\partial y_{10}} \\ \frac{1}{2}\frac{\partial}{\partial y_7} & \frac{1}{2}\frac{\partial}{\partial y_9} & \frac{1}{2}\frac{\partial}{\partial y_{10}} & \frac{\partial}{\partial y_4} \end{pmatrix} \quad \text{and} \quad \frac{\partial}{\partial V} = \begin{pmatrix} \frac{\partial}{\partial v_{11}} & \frac{\partial}{\partial v_{12}} & \frac{\partial}{\partial v_{13}} & \frac{\partial}{\partial v_{14}} \\ \frac{\partial}{\partial v_{21}} & \frac{\partial}{\partial v_{22}} & \frac{\partial}{\partial v_{23}} & \frac{\partial}{\partial v_{24}} \end{pmatrix}.$$

Let

$$D_i = \operatorname{tr}\left(\left(2\,Y\frac{\partial}{\partial Y}\right)^i\right), \quad i = 1, 2, 3, 4$$

and

$$\Omega_{pq}^{(k)} = \left\{ \frac{\partial}{\partial V}\left(2Y\frac{\partial}{\partial Y}\right)^k Y\,{}^t\!\left(\frac{\partial}{\partial V}\right) \right\}_{pq}, \quad k = 0, 1, 2, 3,\ 1 \le p \le q \le 2.$$

Note that D_1, D_2, D_3, D_4, $\Omega_{11}^{(0)}, \ldots, \Omega_{22}^{(3)}$ are $GL_{4,2}$-invariant. It is easily seen that

$$D_1 = \operatorname{tr}\left(2\,Y\frac{\partial}{\partial Y}\right) = 2\sum_{i=1}^{10} y_i\frac{\partial}{\partial y_i}.$$

For brevity, we put

$$\partial_{ij} = \frac{\partial}{\partial v_{ij}}, \quad i = 1, 2,\ 1 \le j \le 4.$$

Then we get

$$\begin{aligned} \Omega_{11}^{(0)} &= y_1\,\partial_{11}^2 + y_2\,\partial_{12}^2 + y_3\,\partial_{13}^2 + y_4\,\partial_{14}^2 + 2\,y_5\,\partial_{11}\partial_{12} + 2\,y_6\,\partial_{11}\partial_{13} \\ &\quad + 2\,y_7\,\partial_{11}\partial_{14} + 2\,y_8\,\partial_{12}\partial_{13} + 2\,y_9\,\partial_{12}\partial_{14} + 2\,y_{10}\,\partial_{13}\partial_{14}, \\ \Omega_{12}^{(0)} &= y_1\,\partial_{11}\partial_{21} + y_2\,\partial_{12}\partial_{22} + y_3\,\partial_{13}\partial_{23} + y_4\,\partial_{14}\partial_{24} \\ &\quad + y_5\left(\partial_{11}\partial_{22} + \partial_{12}\partial_{21}\right) + y_6\left(\partial_{11}\partial_{23} + \partial_{13}\partial_{21}\right) \\ &\quad + y_7\left(\partial_{11}\partial_{24} + \partial_{14}\partial_{21}\right) + y_8\left(\partial_{12}\partial_{23} + \partial_{13}\partial_{22}\right) \end{aligned}$$

$$+\ y_9\left(\partial_{12}\partial_{24}+\partial_{14}\partial_{22}\right)+\ y_{10}\left(\partial_{13}\partial_{24}+\partial_{14}\partial_{23}\right),$$

$$\Omega_{22}^{(0)}=y_1\,\partial_{21}^2+y_2\,\partial_{22}^2+y_3\,\partial_{23}^2+y_4\,\partial_{24}^2+2\,y_5\,\partial_{21}\partial_{22}+2\,y_6\,\partial_{21}\partial_{23}+2\,y_7\,\partial_{21}\partial_{24}+2\,y_8\,\partial_{22}\partial_{23}+2\,y_9\,\partial_{22}\partial_{24}+2\,y_{10}\,\partial_{23}\partial_{24}.$$

Then we have the following relations

$$[D_i,D_j]=0\quad\text{for all } i,j=1,2,3,4, \tag{7.14}$$

$$[\Omega_{kl}^{(0)},\Omega_{pq}^{(0)}]=0,\quad 1\le k\le l\le 2,\ 1\le p\le q\le 2, \tag{7.15}$$

and

$$[D_1,\Omega_{11}^{(0)}]=2\,\Omega_{11}^{(0)},\quad [D_1,\Omega_{12}^{(0)}]=2\,\Omega_{12}^{(0)},\quad [D_1,\Omega_{22}^{(0)}]=2\,\Omega_{22}^{(0)}. \tag{7.16}$$

Therefore, $\mathbb{D}(\mathcal{P}_{4,2})$ is not commutative.

8. Final remarks

In this section, we present some open problems and discuss a notion of automorphic forms on $\mathcal{P}_{n,m}$ using $GL_{n,m}$-invariant differential operators on the Minkowski-Euclid space $\mathcal{P}_{n,m}$.

Recall the invariant polynomials $\alpha_j\,(1\le j\le n)$ from (3.11) and $\beta_{pq}^{(k)}$ $(0\le k\le n-1,\ 1\le p\le q\le m)$ from (3.12). Also recall the invariant differential operators $D_j\,(1\le j\le n)$ from (3.19) and $\Omega_{pq}^{(k)}$ $(0\le k\le n-1,\ 1\le p\le q\le m)$ from (3.20).

Theorem 8.1. *The following relations hold*:

$$[D_i,D_j]=0\quad \textit{for all } 1\le i,j\le n, \tag{8.1}$$

$$[\Omega_{kl}^{(0)},\Omega_{pq}^{(0)}]=0,\quad 1\le k\le l\le m,\ 1\le p\le q\le m, \tag{8.2}$$

and

$$[D_1,\Omega_{pq}^{(0)}]=2\,\Omega_{pq}^{(0)}\quad \textit{for all}\ \ 1\le p\le q\le m. \tag{8.3}$$

Proof. The relation (8.1) follows from the work of Atle Selberg (cf. [8, 10, 11]). Take a coordinate (Y,V) in $\mathcal{P}_{n,m}$ with $Y=(y_{ij})$ and $V=(v_{kl})$. Put

$$\frac{\partial}{\partial Y}=\left(\frac{1+\delta_{ij}}{2}\frac{\partial}{\partial y_{ij}}\right)\quad\text{and}\quad \frac{\partial}{\partial V}=\left(\frac{\partial}{\partial v_{kl}}\right),$$

where $1\le i,j,l\le n$ and $1\le k\le m$. Then we get

$$D_1=2\sum_{1\le i\le j\le n} y_{ij}\,\frac{\partial}{\partial y_{ij}},$$

$$\Omega_{pq}^{(0)}=\sum_{a=1}^{n} y_{aa}\,\frac{\partial^2}{\partial v_{pa}\partial v_{qa}}+\sum_{1\le a<b\le n} y_{ab}\left(\frac{\partial^2}{\partial v_{pa}\partial v_{qb}}+\frac{\partial^2}{\partial v_{pb}\partial v_{qa}}\right).$$

By a direct calculation, we obtain the desired relations (8.2) and (8.3). □

Conjecture 2.

$$(8.4) \qquad \Theta_{n,m}(\alpha_j) = D_j \quad \text{for all } 1 \leq j \leq n,$$

$$(8.5) \qquad \Theta_{n,m}\big(\beta_{pq}^{(k)}\big) = \Omega_{pq}^{(k)} \quad \text{for all } 0 \leq k \leq n-1,\ 1 \leq p \leq q \leq m.$$

We refer to Conjecture 1 in Section 2.

Conjecture 3. The invariants $D_j\,(1 \leq j \leq n)$ and $\Omega_{pq}^{(k)}$ $(0 \leq k \leq n-1,\ 1 \leq p \leq q \leq m)$ generate the noncommutative algebra $\mathbb{D}(\mathcal{P}_{n,m})$.

Conjecture 4. The above relations (8.1), (8.2) and (8.3) generate all relations among the set

$$\Big\{ D_j,\, \Omega_{pq}^{(k)} \mid 1 \leq j \leq n,\ 0 \leq k \leq n-1,\ 1 \leq p \leq q \leq m \Big\}.$$

Problem 8. Find a natural way to construct generators of $\mathbb{D}(\mathcal{P}_{n,m})$.

Using $GL_{n,m}$-invariant differential operators on the Minkowski-Euclid space $\mathcal{P}_{n,m}$, we introduce a notion of automorphic forms on $\mathcal{P}_{n,m}$ (cf. [11]).

Let

$$\Gamma_{n,m} := GL(n,\mathbb{Z}) \ltimes \mathbb{Z}^{(m,n)}$$

be the arithmetic subgroup of $GL_{n,m}$. Let $\mathcal{Z}_{n,m}$ be the center of $\mathbb{D}(\mathcal{P}_{n,m})$.

Definition 8.1. A smooth function $f : \mathcal{P}_{n,m} \longrightarrow \mathbb{C}$ is said to be an automorphic form for $\Gamma_{n,m}$ if it satisfies the following conditions:

(A1) f is $\Gamma_{n,m}$-invariant.

(A2) f is an eigenfunction of any differential operator in the center $\mathcal{Z}_{n,m}$ of $\mathbb{D}(\mathcal{P}_{n,m})$.

(A3) f has a growth condition.

We define another notion of automorphic forms as follows.

Definition 8.2. Let $\mathbb{D}_\spadesuit$ be a commutative subalgebra of $\mathbb{D}(\mathcal{P}_{n,m})$ containing the Laplacian $\Delta_{n,m;A,B}$. A smooth function $f : \mathcal{P}_{n,m} \longrightarrow \mathbb{C}$ is said to be an automorphic form for $\Gamma_{n,m}$ with respect to $\mathbb{D}_\spadesuit$ if it satisfies the following conditions:

(A1) f is $\Gamma_{n,m}$-invariant.

(A2) f is an eigenfunction of any differential operator in $\mathbb{D}_\spadesuit$.

(A3) f has a growth condition.

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